Lesson 12

Binomial Models for American Options

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Lesson Plan

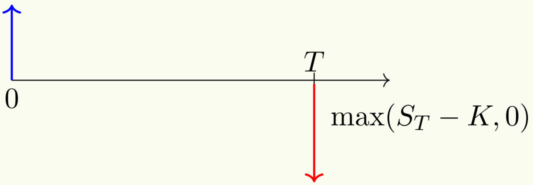
Options in the Real World

- The **exercise style** of listed options are American by default, except for options on equity market indexes such as the S&P 500 index.
- In another words, options on individual stocks and ETFs are exercised in **American style**, whereas most index options are **European style**.
- **Options on futures** are typically American as well.
- The **Black-Scholes pricing formulas** are not applicable on American options.
- Being an algorithm, **binomial option pricing models**, nevertheless, can be modified to take care of the added complication in the American option.

European versus American

European-style

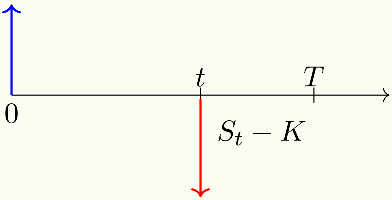
Seller sells the (call) option to allow the buyer to buy the underlying at the price of K on **expiration date** only.



Cash flow of European option seller

American-style

Seller sells the (call) option to allow the buyer to buy the underlying at the price of K and another option to buy at any time no later than the expiration date.



Cash flow of American option seller

Early Exercise Premium

- The option that gives the buyer the flexibility to exercise before expiration is called the **early exercise** option.
- The premium C_t of an American call option may thus be decomposed as

$$C_t = c_t + e_{c,t},$$

where c_t is the corresponding European call option and $e_{c,t}$ is the **early exercise premium**.

- Likewise, for an American put option, its price P_t may be decomposed as

$$P_t = p_t + e_{p,t},$$

where p_t is the corresponding European put option and $e_{p,t}$ is the **early exercise premium**.

Another Model-Free Property of an European Call

- By subtracting and adding the strike price K, the **put-call parity** at time t can be re-written as

$$c_t = S_t - K + p_t + K\left(1 - e^{-r(T-t)}\right). \tag{1}$$

- The first term $S_t - K$ is simply the value that would be obtained if the call option was exercised immediately at time t.
- The last term $K\left(1 - e^{-r(T-t)}\right)$ is intuitively interpreted as the interest amount payable at time t of a continuously compounding loan K.
- Since the put option in (1) and the interest payable are positive numbers (assuming $r > 0$), it must be that

$$c_t > S_t - K.$$

- This inequality is simply the statement that the call option's price is greater than the **intrinsic value**.

When Not to Exercise the American Call Early?

- Thanks to the early exercise premium $e_{c,t}$, the American call option C_t on a **non-dividend-paying stock** costs more than an otherwise identical European call option c_t, i.e.,

$$C_t \geq c_t.$$

- It follows that the early exercise of an American call option is not rational whenever the option premium is higher than $S_t - K$, i.e.,

$$C_t \geq c_t > S_t - K.$$

- Since everybody knows that early exercise of an American call option on non-dividend paying stock does not make sense, buyers will be unwilling to pay for the early exercise premium $e_{c,t}$.
- Consequently, $C_t = c_t$.

Effects of Dividend

- **Ex date**: the date on which stock buyer will not receive the current **dividend** but the seller will.
- If the company's fundamentals do not change, and the entire stock market is quiet, then the stock will have to drop by an amount equal to the dividend.
- Otherwise, there is an **arbitrage opportunity**.
  - Buy the stock one day before the ex date at x dollars
  - If the stock is still selling at x dollars, sell the stock and pocket the dividend.

Q1. What will happen to the call price on ex date?

A1. ______________________________

Q2. What will happen to the put price on ex date?

A2. ______________________________

When to Exercise an American Call Early?

- If the stock is paying a dividend for which the ex date is before the option expiration date, then it *may be* profitable to exercise the American call option early.
- In **practice**, the conditions that make the early exercise decision favorable are as follows:
  1. The option is deep-in-the-money and has a **delta** of $\approx$ 100 (%);
  2. The option has little **time value**;
  3. The dividend is relatively high and its **ex date** precedes the **option expiration date**.

Case Study from Investopedia

- Stock price $S_t = \$100$, pay \$2 dividend and ex-dividend date is tomorrow $(t+1)$.
- You bought an **American call**: C_t(90) = \$10 $\big($your delta is about 100(%)$\big)$
- You have three possible courses of action:

(1) Do nothing and hold the option

Come tomorrow, $S_{t+1} = \$98$, and your option value will drop \$2. You lose \$200. (Options are traded in the units of 100.)

(2) Exercise the option early today

Give up \$10 in C_t(90) and pay the strike price of \$90 per share. Effectively, you buy the stock at \$100 per share. Come tomorrow, S_t = \$98 but you receive \$2 dividend. So you have preserved the value of your portfolio.

Case Study from Investopedia (つづき)

(3) Sell the option and buy 100 shares of the stock

Similar to exercising the option early.

- Suppose the option price is trading at \$11 (per option) instead
  The best course of action is to sell the call and collect \$11 today and buy the stock at \$100 (to preserve your 100 delta position). Come tomorrow, S_{t+1} = \$98 but you will receive \$2 dividend. If you have bought $C(90)$ at \$10, and sell it at \$11, there is a profit of \$1 per option.

- The other two courses of action are not interesting.

- The upshot is that exercising an ITM American call is not profitable even if there is dividend.

When Not to Exercise an American Put Early?

- Suppose the stock does not pay **dividend** before the option **expiration date**. Applying the **put-call parity**, and having added and subtracted K, the European put option is expressed as

$$p_t = K - S_t + c_t - K\left(1 - e^{-r(T-t)}\right).$$

- If the stock does not pay dividend, then $c_t = C_t$, the above **put-call parity** is re-written as

$$p_t - (K - S_t) = C_t - K\left(1 - e^{-r(T-t)}\right).$$

- The term $K - S_t$ is none other than the value or the payoff that the put option holder would get if it was exercised at time t.

When Not to Exercise an American Put Early? (つづき)

- When $C_t > K\left(1 - e^{-r(T-t)}\right)$, it must be that $p_t > K - S_t$. Since $P_t \geq p_t$, it follows that

$$P_t \geq p_t > K - S_t,$$

i.e., the American put option is more valuable than if it were to be exercised immediately to receive the **payoff** $K - S_t$.

- Therefore, it is not profitable to exercise the American put option if its price is higher than the payoff $K - S_t$ under the condition that $C_t > K\left(1 - e^{-r(T-t)}\right)$.
- The put option holder is better off to sell the American option at the price of P_t in the market than to exercise it.

When to Exercise an American Put Early?

- But if $C_t - K\big(1 - e^{-r(T-t)}\big) < 0$ and if S_t is sufficiently below K (at least to break even), then it may become profitable to exercise the American put option early.
- Generically, for any European put option, the lower bound of its price is $\big(Ke^{-r(T-t)} - S_t\big)^+$.
- When the put is in the money, i.e., $S_t < K$,

$$\big(Ke^{-r(T-t)} - S_t\big)^+ < K - S_t,$$

which means that **early exercise** of an American put may become profitable.

European versus American Portfolios

- Portfolio G
  A European call struck at K and expiring on day T, plus cash K invested in a risk-free security yielding r
- Portfolio H
  An American put option struck at K and expiring on day T, plus one share of stock
- We examine the value of portfolio G. At time 0, its value is $c_0 + K$. At expiration time T, portfolio G is worth
  $$\max(S_T - K, 0) + Ke^{rT} = \max(S_T, K) + K\left(e^{rT} - 1\right).$$
  The right side of the above expression is obtained when the following two scenarios are incorporated.

European versus American Portfolios (つづき)

1. The European call option is at or in the money since $S_T \geq K$. Then the value of portfolio G is $S_T - K + Ke^{rT}$, which is $S_T + K\left(e^{rT} - 1\right)$.
2. The European call option is out of the money because $S_T < K$. Then the value of portfolio G is $0 + Ke^{rT}$, which is $K + K\left(e^{rT} - 1\right)$.

- The term common to these two mutually exclusive scenarios is $K\left(e^{rT} - 1\right)$, i.e., the interest earned from time 0 to time T for an initial principal sum of K.
- In the first scenario, the uncommon term is S_T, which is larger than or equal to K. For the second scenario, the uncommon term is K, which is larger than S_T.
- If we write the uncommon term as $\max(S_T, K)$, these two scenarios are thus taken care of correctly and nicely.

European versus American Portfolios (つづき)

- We examine the value of Portfolio H. If the put option is exercised early at time t_e by selling the one share at the strike price, the value of portfolio H will become K.
- Now, the **marked-to-market value** of portfolio G is $c_{t_e} + Ke^{rt_e}$. Also evidently, since $K < Ke^{rt_e}$, it follows that
$$K = V_H(t_e) < V_G(t_e) = c_{t_e} + Ke^{rt_e}.$$
- Notice that $t_e < T$ is a **random time** depending on K, S_t, r, and T, among others.
- In any case, regardless of whether the American put option is exercised or not, V_H is smaller than V_G at time T and at any early exercise time $t_e < T$.

Lower Bound of $C_0 - P_0$

- By the third principle of $\mathbb{Q}$F, the value of portfolio H must also be smaller than portfolio G at time 0.
- Since $c_0 \le C_0$, the value $V_G(0)$ of portfolio G, which is $c_0 + K$, will be smaller or equal to $C_0 + K$. It follows that

$$P_0 + S_0 \le C_0 + K.$$

- In this fashion, we have obtained the lower bound for the synthetic long forward $C_0 - P_0$ as follows:

$$S_0 - K \le C_0 - P_0. \tag{2}$$

- The equality occurs in the trivial case when the options expire ($T = 0$).

Upper Bound of $C_0 - P_0$

- Portfolio I
  An American call struck at K and maturing at time T, plus an amount Ke^{-rT} invested in a risk-free security paying the rate of r
- Portfolio J
  A European put option with the same strike price and expiration, plus one share of the underlying stock
- If the American call is held to maturity, the value of portfolio I will become

$$V_I(T) = \max(S_T - K, 0) + K = \max(S_T, K).$$

- On the other hand, portfolio J's value will also be

$$V_J(T) = \max(K - S_T, 0) + S_T = \max(S_T, K).$$

Upper Bound of $C_0 - P_0$ (つづき)

- Therefore, in the case where the American call option is not exercised early, we have $V_I(T) = V_J(T)$.
- But if the call option is exercised early at any time t_e before T, the value $V_I(t_e)$ of portfolio I becomes
$$S_{t_e} - K + Ke^{-r(T-t_e)} = S_{t_e} - K\Big(1 - e^{-r(T-t_e)}\Big),$$
which is less than S_{t_e}.
- Portfolio J's value when marked to market at time t_e will be $p_{t_e} + S_{t_e}$, which is larger than S_{t_e} and hence also larger than $V_I(t_e)$.
- Therefore, this analysis suggests that $V_I(T) = V_J(T)$ at time T, and $V_I(t_e) < V_J(t_e)$ for any $t_e < T$.

Upper Bound of $C_0 - P_0$ (つづき)

- Because portfolio J is worth at least as much as portfolio I in all circumstances, by the third principle of $\mathbb{Q}$F, at time 0, it must be that

$$V_I(0) = C_0 + Ke^{-rT} \leq p_0 + S_0 = V_J(0).$$

- Since the European put p_0 is less than or equal to the American put P_0, it follows that

$$C_0 + Ke^{-rT} \leq P_0 + S_0,$$

and hence

$$C_0 - P_0 \leq S_0 - Ke^{-rT}.$$

- Notice that if both options were European, this result would be the **put-call parity**.

Put-Call Inequality

- In conjunction with the lower bound result (2) obtained earlier for the American options, the **put-call inequalities** are written as

$$\boxed{S_0 - K \leq C_0 - P_0 \leq S_0 - Ke^{-rT}.} \tag{3}$$

- As a corollary of these inequalities, we have

$$-K\left(1 - e^{-rT}\right) < e_{c,0} - e_{p,0} \leq 0,$$

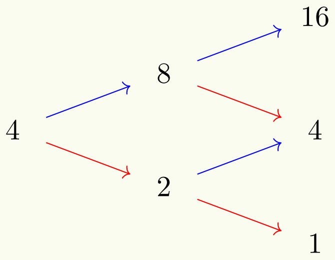
equivalently,

$$0 \leq e_{p,0} - e_{c,0} < K\left(1 - e^{-rT}\right).$$

- Everything else being equal, the early exercise premium of an American put option is larger and no less than that of a corresponding American call option.

Revision: Binomial Tree Algorithm

- Two-step **binomial tree** given by the parameters:
  - $S_0 = 4$
  - $u = 2$
  - $d = 1/2$
  - $r = 22.31\%$ (artificially made very large to get nice numbers)
  - $\Delta t = 1$

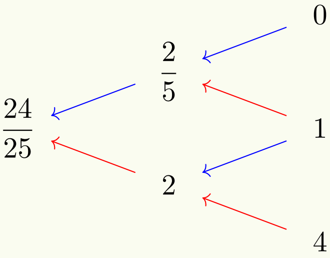


Binomial Tree for European Put

- Compute the **risk-neutral probability** of upward movement p, and set $q := 1 - p$.
- To value a European put option struck at $K = 5$, we evaluate

$$V_n = e^{-r\Delta t}\mathbb{E}_n^{\mathbb{Q}}\left(V_{n+1}\right) = e^{-r\Delta t}\left(pV_{n+1}^{+} + qV_{n+1}^{-}\right).$$

- The result is $V_0 = p_0 = \dfrac{24}{25}$

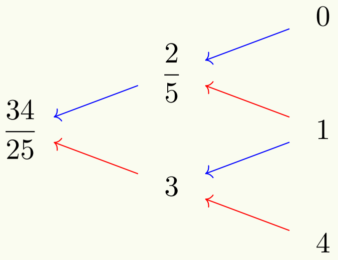


Binomial Tree for American Put

- At each time step prior to the expiry nodes, the early exercise provision in the American option gives you the choice of either to exercise immediately and receive the **intrinsic value** of the option, or to hold on to the option to the next step.

$$V_n = \max\left(e^{-r\Delta t}\left[pV_{n+1}^{+} + qV_{n+1}^{-}\right], (K - S_n)^{+}\right).$$

- Continuing from the earlier example in Slide 23,



Concept Checkers

- What is the early exercise premium of the American put in Slide 25?

  Answer: ____________

- What is the upper bound for the corresponding American call?

  Answer: ____________

- What is the lower bound for the corresponding American call?

  Answer: ____________

Takeaways

- In reality, less than 10% of the American options are exercised.
- The early exercise feature in the American options makes it very difficult to price the option premiums.
- If an American call option expires before a dividend ex date, it is never rational to exercise the option early.
- For American options, instead of put-call parity, we have put-call inequality.
- The binomial tree is a popular algorithm to price American options.

Keywords