

Lesson 10

Derivatives of Non-Linear Payoffs

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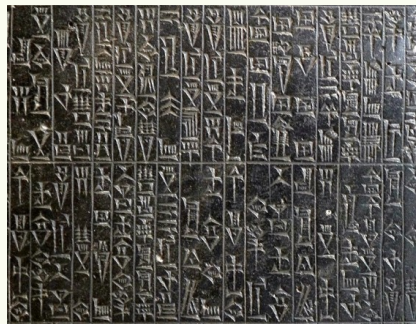
Broad Lesson Plan

- 1 はじめに
- 2 Options' Basics
- 3 Put-Call Parity
- 4 Overall Shape of Call
- 5 Box Spread Parity
- 6 持ち帰り

Code of King Hammurabi, 1792 to 1750 BC



Picture source: [The Louvre](#)



Picture source: [Code of Hammurabi](#)

King Hammurabi's 48-th Code

If any one owe a debt for a loan, and a storm prostrates the grain, or the harvest fail, or the grain does not grow for lack of water; in that year he need not give his creditor any grain, he washes his debt-tablet in water and pays no rent for this year.

From the perspective of a creditor,

- ⇒ **Underlying asset**: grain
- ⇒ **Expiration**: at harvest
- ⇒ **Delivery mode**: physical
- ⇒ **Contingent condition**: if not (a storm prostrates the grain, or the harvest fail, or the grain does not grow for lack of water) deliver x tons of grain

Publicly Listed Options

- Options used to be, and still is traded OTC.
- Chicago Board of Options Exchange (**CBOE**) for publicly listed options
- Black-Scholes' pricing formulas
- **OCC Market Data**
 - 1973: 1.1 million option contracts on 32 equity issues
 - 2023: 10.1 billion contracts on 5,894 issues

Basic Option Terminology

- ↔ **Strike price** K : the equivalent of forward price in the forward contract
- ↔ **Plain vanilla**: standard type of option, one with a simple expiration date and strike price and no additional feature
- ↔ **Call option**: **right** (**obligation**) of an option buyer (seller) to buy (sell) the underlying asset at the strike price
- ↔ **Put option**: right (obligation) of an option buyer (seller) to sell (buy) the underlying asset at the strike price
- ↔ **European**: buyers can exercise the right to buy or sell at expiration date only
- ↔ **American**: buyers can exercise the right to buy or sell at or before expiration

Concept Checker: Odd One Out?

- 1 For European option buyers, they must hold the option to maturity.
- 2 For European option sellers, they must hold the option to maturity.
- 3 At expiration, American options cannot be exercised.
- 4 Any option seller can exit the obligation by buying back the options sold in the marketplace.

Moneyiness of Call Options

- ↔ **In-the-money call** option: strike price is lower than the underlying price
- ↔ **At-the-money call** option: strike price is equal to the underlying price
- ↔ **Out-of-the-money call** option: strike price is higher than the underlying price

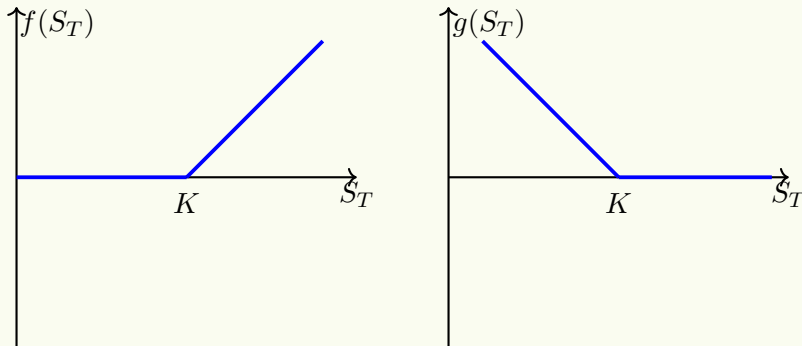
Moneyness of Put Options

- ↔ **In-the-money put** option: strike price is higher than the underlying price
- ↔ **At-the-money put** option: strike price is equal to the underlying price
- ↔ **Out-of-the-money put** option: strike price is lower than the underlying price
- ↔ **DOOM** option: Deep out-of-the-money option

Option Price

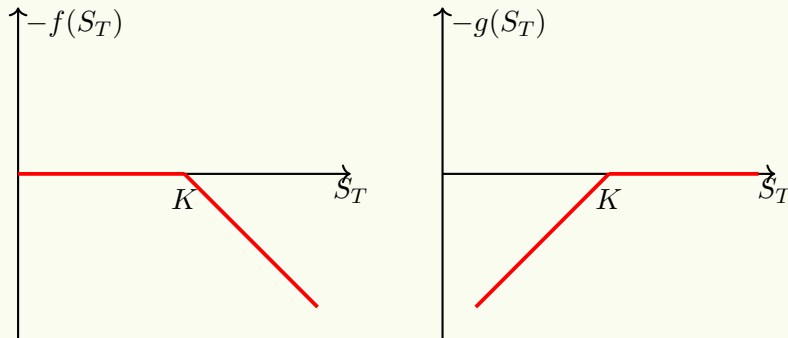
- ↔ Because of the asymmetry of right and obligation, option sellers demand a **premium** from option buyer.
- ↔ The premium is also known as the **option price**.
- ↔ One option contract in the U.S. typically allows the call (put) option buyer to exercise the right to buy (sell) 100 shares at the strike price K .
- ↔ But how should one go about pricing an option?

Buyers' Payoff Functions of Plain Vanilla Options



The payoffs of plain vanilla call (left) and put (right) options buyer at maturity T when the underlying price is S_T .

Sellers' Payoff Functions of Plain Vanilla Options



The payoffs of plain vanilla call (left) and put (right) options seller at maturity T when the underlying price is S_T .

Quiz: Learn to Think Geometrically

(1) What is the value of $f(S_T) - g(S_T)$?

Answer: _____

(2) What is the value of $g(S_T) - f(S_T)$?

Answer: _____

(3) What is the functional form of call's payoff?

Answer: _____

(4) What is the functional form of put's payoff?

Answer: _____

Option Chain (of Strike Prices)

Underlying stock: **Citigroup Inc. (C)** Last: \$52.84 Change: \$1.32 %Change: 2.56% as of Jan 22 2024, 4:00 PM ET **Options Expiration: Jan 17, 2025.**

Call						Put						
LAST	CHG	BID	ASK	VOL	OPEN INT.	STRIKE	LAST	CHG	BID	ASK	VOL	OPEN INT.
0	0	31.3	34.4	0		20	0.15	0.04	0.1	0.11	4	1,577
28.75	0	29.05	31.2	0	58	22.5	0.13	-0.04	0.12	0.15	5	2,559
27.75	1.7	26.75	29.05	9	256	25	0.23	-0.04	0.18	0.21	2	5,930
15.4	0	24.35	26.45	0	99	27.5	0.32	0	0.26	0.29	1	11,637
23.18	1.93	22.65	23.45	13	1,176	30	0.38	-0.05	0.35	0.4	105	10,350
19.75	0	19.75	21.9	0	506	32.5	0.51	-0.15	0.5	0.52	12	11,339
18.35	1.35	18	18.65	1	1,424	35	0.69	-0.19	0.68	0.7	232	12,895
17.1	0	15.2	16.4	0	11,905	37.5	0.92	-0.19	0.9	0.94	220	30,068
14.15	1.15	13.8	14.4	156	8,875	40	1.24	-0.2	1.21	1.25	212	26,743
12.09	0.99	11.8	12.15	1,423	20,089	42.5	1.7	-0.22	1.62	1.67	6	26,574
10.2	1.45	10	10.3	67	39,060	45	2.2	-0.34	2.15	2.22	285	69,623
8.59	0.99	8.35	8.6	10	24,168	47.5	3.04	-0.28	2.86	2.93	38	16,890
7	0.85	6.95	7.05	44	47,202	50	3.8	-0.52	3.7	4.35	3,581	36,719
5.7	0.78	5.6	5.7	2,772	56,300	52.5	4.87	-0.73	4.8	4.95	238	12,019
Current price as of Jan 22, 2024 4:00 p.m. EST						\$52.84						
4.55	0.64	4.5	4.6	456	32,159	55	6.15	-0.75	6.1	6.2	41	13,521
3.64	0.62	3.5	3.65	759	14,290	57.5	7.7	-0.65	7.1	7.75	1	1,880
2.84	0.44	2.84	2.86	270	51,580	60	9.46	-0.89	9.25	9.45	2	4,122
2.19	0.4	2.21	2.24	509	1,333	62.5	11.27	-1.63	11.1	11.75	1	195
1.72	0.3	1.67	1.74	459	66,684	65	14.45	0.15	11.7	14.25	46	430
1.01	0.13	0.98	1.06	960	46,262	70	19.27	2.07	17.45	17.85	1	258
0.62	0.08	0.6	0.65	243	20,765	75	23.9	2.35	22.15	23.1	1	16
0.4	0.07	0.38	0.41	20	836	80	0	0	26.15	29.1	0	

Option Market Quotes and Volume

- ↔ Bid: the price market is currently willing to pay for buying an option
- ↔ Ask: the price at which market will sell an option
- ↔ Last: the price at which an option was traded most recently
- ↔ Volume: number of contracts traded during the trading session
- ↔ Open Interest: number of open contracts

Intrinsic Value

- ↔ The dollar amount an option is in the money.
- ↔ Out-of-the-money options have no **intrinsic value**.
- ↔ What about at-the-money options?

Answer: _____

- ↔ Example: Strike price \$52.50 versus underlying price of \$52.84.
 - Intrinsic value of call option = $\$52.84 - \$52.50 = \$0.34$.
 - Intrinsic value of put option = 0

Time Value

- ↔ For the call option, the intrinsic value is only \$0.34 whereas the midpoint of its bid and ask prices is \$5.65. Why is the call option priced at a price higher than the intrinsic value?
- ↔ The difference of $\$5.65 - \$0.34 = \$5.31$ is called the **option's time value**.
- ↔ Why would a call option buyer pay for the time value?
- ↔ For that matter, why would the put option of zero intrinsic value is worth $(\$4.80 + \$4.95)/2 = \$4.875$?

Concept Checker: Odd One Out?

- 1 Option holders have rights but sellers have obligations.
- 2 Only in-the-money options have intrinsic value.
- 3 The time value at expiration is zero.
- 4 The time value of option is always bigger than the intrinsic value.

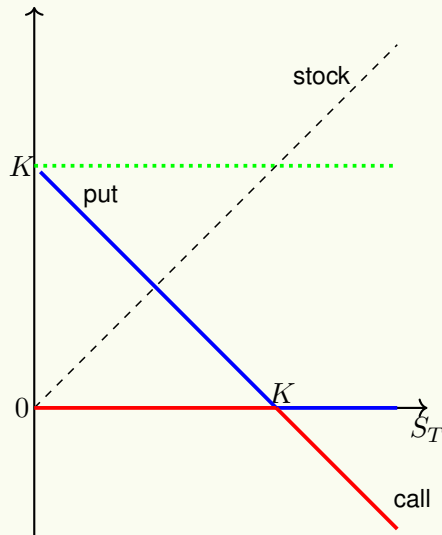
Put and Call

- ➡ Consider a put p_t and a call c_t on the same underlying asset S_t , maturity T , and strike K . Of these 5 quantities, only T is not a price.
- ➡ In addition, suppose lending and borrowing can be done at the interest rate of r_0 for tenor T with continuous compounding
- ➡ Positions at time $t = 0$
 - Borrow an amount Ke^{-r_0T}
 - Sell a call c_0
 - Use the borrowed fund and sales proceeds to buy the stock at price S_0 , and a put p_t .
- ➡ Total cash flow at time 0: $Ke^{-r_0T} + c_0 - S_0 - p_0$

Payoffs of Long Stock, Short Call, and Long Put

- ➡ At maturity $t = T$, if $S_T < K$, then
- The call option is worthless.
 - Exercise the put option to sell the asset at the strike price K .
 - Pay the debt of amount K
- ➡ If $S_T > K$, then
- The put option is worthless.
 - The call option holder will exercise and you have to sell the asset at the strike price K .
 - Pay the debt of amount K
- ➡ If $S_T = K$, then
- Both options are not exercised
 - Selling the underlying asset at S_T in the spot market.
 - Pay the debt of amount K

Payoff Diagram at Expiration



Application of the Principles of QF

- ➡ The net cash flow at time T is zero, regardless of the outcomes (either $S_T < K$ or $S_T > K$ or $S_T = K$).
- ➡ By the first principle of QF, the cash flow at time 0 must also be zero because there is no uncertainty and hence no risk. Why no uncertainty? All the prices and the interest rate are known at time 0!
- ➡ Hence

$$Ke^{-rT} + c_0 - S_0 - p_0 = 0.$$

and this **put-call parity** is more commonly written as

$$c_0 - p_0 = S_0 - Ke^{-r_0T}.$$

- ➡ At time t , it is written as

$$c_t - p_t = S_t - Ke^{-r_t(T-t)}$$

(1)

First Application of Put-Call Parity

- ➡ With respect to the intersection point, the put-call parity (1) becomes

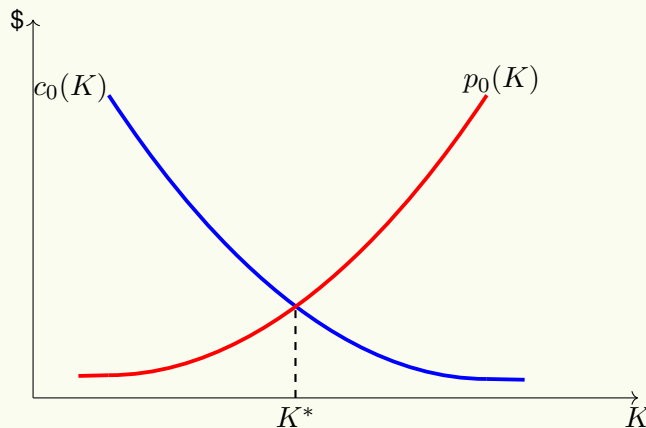
$$0 = S_t - K^* e^{-r_t(T-t)},$$

and it is re-written as

$$K^* = S_t e^{r_t(T-t)}.$$

- ➡ This result shows that the strike price K^* at which the call option price equals the put option price is none other than the **forward price** of the underlying asset at time t .
- ➡ K^* is called the **implied forward price**.

Option Price Curves as Functions of Strike K



Capital Structure

- ➡ The composition of equity and bond by which a corporation finances its assets is called the **capital structure**.
- ➡ A firm that has zero debt is said to be **unlevered**, whereas a firm that also issues corporate bond in addition to equity is said to be **levered**.
- ➡ Suppose a levered firm has outstanding only a zero-coupon bond with face value K maturing at time T .
- ➡ Next, we denote the market value of an unlevered but otherwise identical firm by S_t .

Modigliani-Miller Proposition I

- ➡ It does not matter what capital structure a company uses to finance the operation.
- ➡ A deeper look at the root of the Modigliani-Miller Proposition I finds that it is grounded on the third principle of QF.
- ➡ If a firm's market value could be changed by changing the proportion of stocks and bonds they issue, then arbitrageurs could also repackage the existing stocks and bonds to make a sure profit.
- ➡ Hence, the value of the firm should depend only on the sum of the values of its stocks and bonds, not on whether the firm's capital is weighted more heavily to debt or equity.

2nd Application: Derivation of MM Proposition I

- ➡ The capital structure of a firm, i.e., whether levered or unlevered, is irrelevant to the market valuation of a firm.
- ➡ The **put-call parity** (1) can also be expressed as, at time 0,

$$S_0 = c_0 + Ke^{-r_0T} - p_0. \quad (2)$$

- ➡ The actual market value of the debt need to be discounted to reflect credit risk so that it is worth only $Ke^{-r_0T} - p_0$.
- ➡ The put premium p_0 may be interpreted as the insurance premium required by the bondholders against default by the levered firm.
- ➡ What about the interpretation of c_0 ?

Tutorial

- ➡ Define an interest rate $\hat{r}$ by

$$Ke^{-\hat{r}_0 T} := Ke^{-r_0 T} - p_0,$$

Show that the **credit spread** $\hat{r}_0 - r_0$ can be well approximated by

$$\hat{r}_0 - r_0 = \frac{p_0}{KT}$$

- ➡ Proof

We rewrite the definition as

$$K(e^{-r_0 T} - e^{-\hat{r}_0 T}) = p_0$$

At the first order of the expansion, we have

$$-r_0 T - (-\hat{r}_0 T) = \frac{p_0}{K} \quad \implies \quad \hat{r}_0 - r_0 = \frac{p_0}{KT}$$

Monotonous with Respect to the Strike Price

- Consider two strike prices K_1 and K_2 with $K_1 < K_2$, and also the following two portfolios:
 - Portfolio 1
A European call option $c_0(K_1)$ of strike K_1
 - Portfolio 2
A European call option $c_0(K_2)$ of strike K_2
- At expiration, the value of portfolio 1 (denoted by $V_1(T)$) is by definition $\max(S_T - K_1, 0)$, whereas portfolio 2's value (denoted by $V_2(T)$) is $\max(S_T - K_2, 0)$.
- With respect to these two strike prices, and the underlying asset price at time T , denoted by S_T , three mutually exclusive scenarios are exhaustive:

1 $S_T < K_1$

2 $K_1 \leq S_T \leq K_2$

3 $S_T > K_2$.

Monotonous with Respect to the Strike Price (つづき)

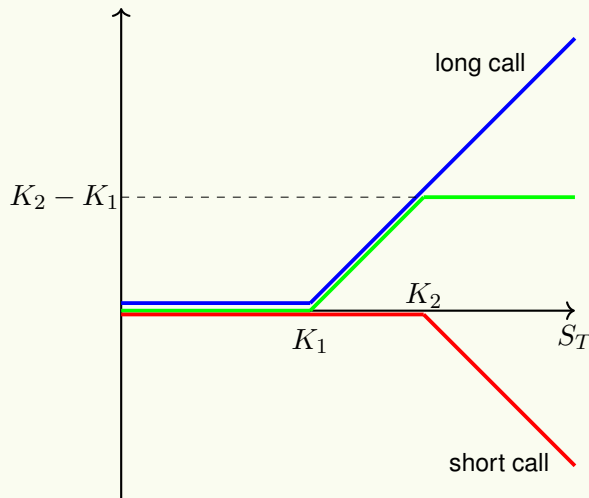
- If $S_T < K_1$, both portfolios are worthless, and $V_2(T) = V_1(T) = 0$.
- If $K_1 \leq S_T \leq K_2$, the value of portfolio 1 is $S_T - K_1$ but the value of portfolio 2 is still zero. So $V_2(T) < V_1(T)$ in the second scenario.
- In the third scenario, both options are in the money, $V_1(T) = S_T - K_1$ and $V_2(T) = S_T - K_2$. Since $K_1 < K_2$, it is clear that $S_T - K_2 < S_T - K_1$, which implies that $V_2(T) < V_1(T)$ in the third scenario as well.
- Accordingly, $V_2(T) \leq V_1(T)$ at time T . By the third principle of $\mathbb{Q}F$, it must be that at time 0,

$$c_0(K_2) \leq c_0(K_1).$$

Lower Bound for the Slope

- What is the lower bound for the slope of the call option price curve $c_0(K)$?
- Portfolio 3: **Bull Call Spread**
A long position in call option $c_0(K_1)$ of strike price K_1 and a short position in call option $c_0(K_2)$ of strike price K_2
- Portfolio 4
A risk-free time deposit of amount $(K_2 - K_1)e^{-rT}$, tenor T , and interest rate r
- The marked-to-market value of portfolio 3 at time 0 is $V_3(0) = c_0(K_1) - c_0(K_2)$, which is non-negative as proven earlier.

Bull Call Spread's Payoff Diagram



Bounds for Call Option's Gradient

- At expiration, it is evident that the payoff of the bull call spread is less than or equal to $K_2 - K_1$, which coincides with the value of portfolio 4.
- Since $V_3(T) \leq V_4(T)$ at time T , their prices at time 0, according to the third principle of $\mathbb{Q}F$, must observe the same inequality:

$$c_0(K_1) - c_0(K_2) \leq (K_2 - K_1)e^{-rT} \leq K_2 - K_1.$$

- In conjunction with the earlier upper bound result that $c_0(K_2) - c_0(K_1) \leq 0$, these inequalities provide an upper bound and a lower bound for the gradient of the call price curve $c_0(K)$ as follows:

$$-1 \leq \frac{c_0(K_2) - c_0(K_1)}{K_2 - K_1} \leq 0.$$

In summary, $c_0(K)$ is a downward sloping curve.

Linear Interpolation

- Consider three strike prices sorted by magnitude as $K_1 < K_2 < K_3$.
- Moreover, we define a positive ratio as follows:

$$\lambda := \frac{K_3 - K_2}{K_3 - K_1} < 1. \quad (3)$$

- This ratio is smaller than one and thereby allows us to express K_2 as a linear combination of K_1 and K_3 :

$$K_2 = \lambda K_1 + (1 - \lambda) K_3.$$

Portfolios for Convexity Analysis

➡ Portfolio 5
A long position in one contract of call option struck at K_2

➡ Portfolio 6
A long position in λ contracts of call option struck at K_1 and another long position in $1 - \lambda$ contracts of call option of strike price K_3

Payoff Scenarios

Scenario	$S_T < K_1$	$K_1 \leq S_T \leq K_2$	$K_2 < S_T \leq K_3$	$S_T > K_3$
Portfolio 5 ◆ Long $1 c_0(K_2)$	0	0	$S_T - K_2$	$S_T - K_2$
Portfolio 6 ◆ Long $\lambda c_0(K_1)$ ◆ Long $1 - \lambda c_0(K_3)$	0 0	$\lambda(S_T - K_1)$ 0	$\lambda(S_T - K_1)$ 0	$\lambda(S_T - K_1)$ $(1 - \lambda)(S_T - K_3)$
Aggregate of Portfolio 6	0	$\lambda(S_T - K_1)$	$\lambda(S_T - K_1)$	$S_T - K_2$

Clearly, at maturity, $V_5(T) \leq V_6(T)$ in all scenarios.

Curvature of the Call Option Price Function

- Again, the third principle of QF requires the prices of these two portfolios at time 0 to be

$$c_0(K_2) \leq \lambda c_0(K_1) + (1 - \lambda)c_0(K_3).$$

- Substituting in the explicit form of λ , i.e., (3) into this inequality, we arrive at

$$(K_3 - K_1)c_0(K_2) \leq (K_3 - K_2)c_0(K_1) + (K_2 - K_1)c_0(K_3).$$

- We write the left side of the inequality as $(K_3 - K_2)c_0(K_2) + (K_2 - K_1)c_0(K_2)$, and then rearrange the inequality into

$$(K_3 - K_2)(c_0(K_2) - c_0(K_1)) \leq (K_2 - K_1)(c_0(K_3) - c_0(K_2)).$$

- Next, we divide both sides by $(K_3 - K_2)(K_2 - K_1)$ to obtain

$$\frac{c_0(K_2) - c_0(K_1)}{K_2 - K_1} \leq \frac{c_0(K_3) - c_0(K_2)}{K_3 - K_2}.$$

Monotonicity, Gradient Boundedness, Convexity

➡ $K_1 < K_2 < K_3$

➡ **Monotonicity** in the option price level

$$c_0(K_2) \leq c_0(K_1); \quad p_0(K_1) \leq p_0(K_2).$$

➡ **Boundedness in the gradient**

$$-1 \leq \frac{c_0(K_2) - c_0(K_1)}{K_2 - K_1} \leq 0; \quad 0 \leq \frac{p_0(K_2) - p_0(K_1)}{K_2 - K_1} \leq 1.$$

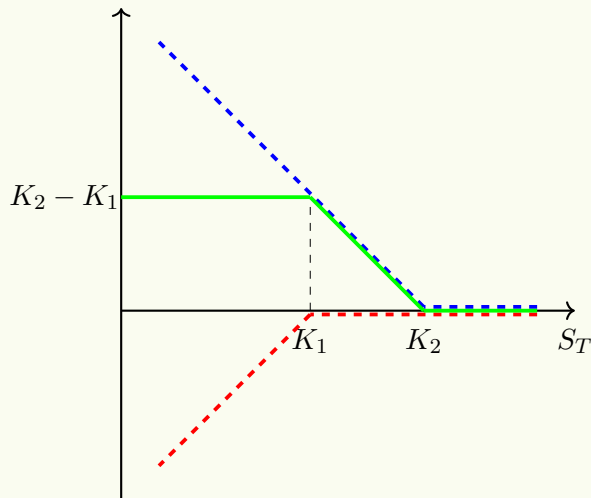
➡ **Convexity**

$$\frac{c_0(K_2) - c_0(K_1)}{K_2 - K_1} \leq \frac{c_0(K_3) - c_0(K_2)}{K_3 - K_2}; \quad \frac{p_0(K_2) - p_0(K_1)}{K_2 - K_1} \leq \frac{p_0(K_3) - p_0(K_2)}{K_3 - K_2}.$$

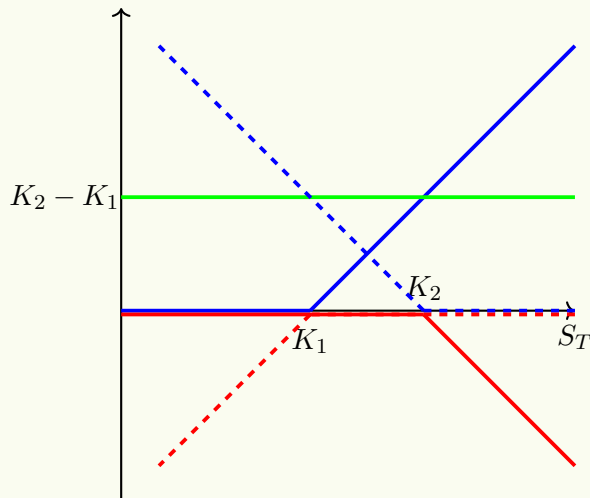
Model-Free

- ➡ It is important to recognize that the three features of monotonicity, gradient boundedness, and convexity are model-free in the following sense:
- Type 1: Models for the stochastic dynamics of the underlying asset are not needed.
 - Type 2: Models to price the options are not needed.
- ➡ Is put-call parity **model-free**? Which Type?

Payoff Diagram of a Bear Put Spread



Payoff Diagram of a Box Spread



Box Spread Cash Flows

Asset price range	Payoff from bull call spread	Payoff from bear put spread	Total payoff
$S_T < K_1$	0	$K_2 - K_1$	$K_2 - K_1$
$K_1 \leq S_T \leq K_2$	$S_T - K_1$	$K_2 - S_T$	$K_2 - K_1$
$S_T > K_2$	$K_2 - K_1$	0	$K_2 - K_1$

→ The payoff of a long position in the bull call spread and a long position in the bear put spread is a constant $K_2 - K_1$!

→ By the first principle of QF, the **present value** of a long position in the box spread must be

$$(K_2 - K_1)e^{-r_0T}.$$

→ Accordingly, we have

$$c_0(K_1) - c_0(K_2) + p_0(K_2) - p_0(K_1) = (K_2 - K_1)e^{-r_0T}.$$

Takeaways

- ◇ Plain vanilla put and call options: exercise styles, maturities, strike prices, settlement modes
- ◇ Intrinsic value versus time value
- ◇ Model-free put-call parity
- ◇ Modigliani-Miller Proposition I: Irrelevance of capital structure of a firm
- ◇ Monotonicity, gradient boundedness, and convexity of option price curves
- ◇ Bull call spread versus bear put spread

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American, 6

At-the-money call, 8

At-the-money put, 9

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