

Lesson 7

Portfolio Optimization

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Inspiring Quote

A good portfolio is more than a long list of good stocks and bonds. It is a balanced whole, providing the investor with protections and opportunities with respect to a wide range of contingencies.

— **Harry Markowitz**

Learning Outcomes

- 👉 Define and describe what a vector derivative is and how it works.
- 👉 Obtain a quick glimpse into partial differentiation.
- 👉 Analyze the component details of matrix calculus.
- 👉 Formulate a Lagrangian for constrained optimization with Lagrange multipliers.
- 👉 Derive the first-order conditions by applying the matrix calculus.
- 👉 Solve the first-order conditions using matrix algebra.
- 👉 Discuss the efficient frontier.

Definition

□ Let \mathbf{x} be a column k -vector. Consider the function

$$g(\mathbf{x}) = g(x_1, x_2, \dots, x_k) : \mathfrak{R}^k \longrightarrow \mathfrak{R}.$$

□ The **vector derivative** is defined as

$$\frac{\partial}{\partial \mathbf{x}} g(\mathbf{x}) = \begin{bmatrix} \frac{\partial}{\partial x_1} g(\mathbf{x}) \\ \frac{\partial}{\partial x_2} g(\mathbf{x}) \\ \vdots \\ \frac{\partial}{\partial x_k} g(\mathbf{x}) \end{bmatrix} \quad \text{and} \quad \frac{\partial}{\partial \mathbf{x}^\top} g(\mathbf{x}) = \left[\frac{\partial}{\partial x_1} g(\mathbf{x}) \quad \frac{\partial}{\partial x_2} g(\mathbf{x}) \quad \dots \quad \frac{\partial}{\partial x_k} g(\mathbf{x}) \right].$$

Basic Properties

□ For constant vector \mathbf{a} and matrix \mathbf{A} ,

$$\frac{\partial}{\partial \mathbf{x}} (\mathbf{a}^\top \mathbf{x}) = \frac{\partial}{\partial \mathbf{x}} (\mathbf{x}^\top \mathbf{a}) = \mathbf{a}, \quad \frac{\partial}{\partial \mathbf{x}^\top} (\mathbf{a}^\top \mathbf{x}) = \mathbf{a}^\top$$

$$\frac{\partial}{\partial \mathbf{x}^\top} (\mathbf{A}\mathbf{x}) = \mathbf{A}$$

$$\frac{\partial}{\partial \mathbf{x}} (\mathbf{x}^\top \mathbf{A}\mathbf{x}) = (\mathbf{A} + \mathbf{A}^\top) \mathbf{x}$$

$$\frac{\partial^2}{\partial \mathbf{x} \partial \mathbf{x}^\top} (\mathbf{x}^\top \mathbf{A}\mathbf{x}) = \mathbf{A} + \mathbf{A}^\top$$

Illustration

$$\square f(x_1, x_2) := a_1x_1 + a_2x_2 = \mathbf{a}^\top \mathbf{x}$$

$$\square \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \mathbf{A}\mathbf{x}.$$

$$\frac{\partial}{\partial \mathbf{x}^\top} (\mathbf{A}\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \mathbf{A}.$$

$$\square g(x_1, x_2) := \mathbf{x}^\top \mathbf{A}\mathbf{x} = a_{11}x_1^2 + a_{22}x_2^2 + a_{12}x_1x_2 + a_{21}x_1x_2$$

$$\frac{\partial}{\partial \mathbf{x}} (\mathbf{x}^\top \mathbf{A}\mathbf{x}) = \begin{bmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2a_{11}x_1 + a_{12}x_2 + a_{21}x_2 \\ 2a_{22}x_2 + a_{12}x_1 + a_{21}x_1 \end{bmatrix} = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

An Interesting Theorem of Matrix Calculation

Theorem 2.1 (Derivative of Trace).

Given two matrices $\mathbf{X}_{p \times q}$ and $\mathbf{A}_{q \times p}$. Then

$$\frac{\partial \text{Tr}(\mathbf{X}\mathbf{A})}{\partial \mathbf{X}} = \mathbf{A}^\top.$$

□ The product of $\mathbf{X}\mathbf{A}$ is

$$\mathbf{X}\mathbf{A} = \begin{bmatrix} x_{11} & \cdots & x_{1q} \\ \vdots & \ddots & \vdots \\ x_{p1} & \cdots & x_{pq} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{q1} & \cdots & a_{qp} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^q x_{1i}a_{i1} & \cdots & \sum_{i=1}^q x_{1i}a_{ip} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^q x_{pi}a_{i1} & \cdots & \sum_{i=1}^q x_{pi}a_{ip} \end{bmatrix}$$

□ The **trace** of $\mathbf{X}\mathbf{A}$ is the sum of the **diagonal elements**.

$$\text{Tr}(\mathbf{X}\mathbf{A}) = \sum_{i=1}^q x_{1i}a_{i1} + \cdots + \sum_{i=1}^q x_{pi}a_{ip} = \sum_{i=1}^q \sum_{j=1}^p x_{ji}a_{ij}$$

An Interesting Theorem of Matrix Calculation (Cont'd)

□ Differentiate a function $f(\mathbf{X})$ with respect to \mathbf{X} is

$$\frac{\partial f(\mathbf{X})}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial f(\mathbf{X})}{\partial x_{11}} & \cdots & \frac{\partial f(\mathbf{X})}{\partial x_{1q}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f(\mathbf{X})}{\partial x_{p1}} & \cdots & \frac{\partial f(\mathbf{X})}{\partial x_{pq}} \end{bmatrix}$$

□ Since $f(\mathbf{X}) = \sum_{i=1}^q \sum_{j=1}^p x_{ji} a_{ij}$, computation shows that

$$\frac{\partial \text{Tr}(\mathbf{X}\mathbf{A})}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial(\sum_{i=1}^q \sum_{j=1}^p x_{ji} a_{ij})}{\partial x_{11}} & \cdots & \frac{\partial(\sum_{i=1}^q \sum_{j=1}^p x_{ji} a_{ij})}{\partial x_{1q}} \\ \vdots & \ddots & \vdots \\ \frac{\partial(\sum_{i=1}^q \sum_{j=1}^p x_{ji} a_{ij})}{\partial x_{p1}} & \cdots & \frac{\partial(\sum_{i=1}^q \sum_{j=1}^p x_{ji} a_{ij})}{\partial x_{pq}} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{q1} \\ \vdots & \ddots & \vdots \\ a_{1p} & \cdots & a_{qp} \end{bmatrix} = \mathbf{A}^\top$$

□

Investment

∞ For each dollar invested, a fraction w_i is invested in **asset** i . It must be that

$$\sum_{i=1}^n w_i = 1.$$

∞ The **weights** are arranged as a n -vector \mathbf{w} .

∞ The **portfolio's expected return** and **expected variance** are, respectively,

$$\mu_p := \mathbb{E}(r_p) = \sum_{i=1}^n w_i \mathbb{E}(r_i) = \sum_{i=1}^n w_i \mu_i = \mathbf{w}^\top \boldsymbol{\mu};$$

$$\sigma_p^2 := \mathbb{V}(r_p) = \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij} = \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w}.$$

Numerical Illustration

- Suppose there are two assets. $\mu_1 = 5\%$ and $\mu_2 = 8\%$ **per annum**.
- The **covariance** is a 2 by 2 matrix.

$$\Sigma := \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 0.0625 & -0.01 \\ -0.01 & 0.16 \end{bmatrix}$$

- Given that the variance of **asset 1** is 0.0625, its **volatility** is $\sqrt{0.0625} = 25\%$ per annum.
- The volatility of asset 2 is _____.
- The **portfolio's expected return** and **expected variance** are, respectively,

$$\mu_p = 0.05w_1 + 0.08w_2$$

$$\sigma_p^2 = 0.0625w_1^2 - 0.01w_1w_2 - 0.01w_2w_1 + 0.16w_2^2$$

Optimization with Constraints

◇ Minimize half the **portfolio variance** under two **constraints**:

$$\sum_{i=1}^n w_i \mathbb{E}(r_i) = \mathbb{E}(r_p).$$

$$\sum_{i=1}^n w_i = 1.$$

◇ **Constrained optimization** with **Lagrange multipliers** λ and ψ :

$$\min_{w_1, w_2, \dots, w_n} L = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij} - \lambda \left(\sum_{i=1}^n w_i \mu_i - \mu_p \right) - \psi \left(\sum_{i=1}^n w_i - 1 \right)$$

In Matrix Form

◇ The **Lagrangian** L is

$$L = \frac{1}{2} \mathbf{w}^\top \Sigma \mathbf{w} - \lambda (\mathbf{w}^\top \boldsymbol{\mu} - \mu_p) - \psi (\mathbf{w}^\top \mathbf{1} - 1).$$

◇ Note that Σ is a **symmetric matrix**.

◇ The **first-order conditions (FOC)** with respect to \mathbf{w} , λ , and ψ are

$$\Sigma \mathbf{w} - \lambda \boldsymbol{\mu} - \psi \mathbf{1} = \mathbf{0}$$

$$\mathbf{w}^\top \boldsymbol{\mu} = \mu_p$$

$$\mathbf{w}^\top \mathbf{1} = 1$$

Solution of First FOC

- ◇ The first FOC gives the solution for the **weight vector**:

$$\mathbf{w}^* = \Sigma^{-1}(\lambda\boldsymbol{\mu} + \psi\mathbf{1}).$$

- ◇ But what are the values of the **Lagrange multipliers** λ and ψ ?

- ◇ To solve for λ and ψ , substitute the **optimal weight vector** above into the last two FOC's,

$$\boldsymbol{\mu}^\top \mathbf{w}^* = \boldsymbol{\mu}^\top \Sigma^{-1}(\lambda\boldsymbol{\mu} + \psi\mathbf{1}) = \mu_p$$

$$\mathbf{1}^\top \mathbf{w}^* = \mathbf{1}^\top \Sigma^{-1}(\lambda\boldsymbol{\mu} + \psi\mathbf{1}) = 1$$

Solution of Second and Third FOCs

◇ Let

$$a := \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$$

$$b := \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}$$

$$c := \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}$$

◇ The last two FOCs can be written as

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \lambda \\ \psi \end{bmatrix} = \begin{bmatrix} \mu_p \\ 1 \end{bmatrix}.$$

◇ Solving these two linear equations, we obtain

$$\lambda = \frac{c\mu_p - b}{ac - b^2}, \quad \psi = \frac{a - b\mu_p}{ac - b^2}.$$

Optimal Weight Vector and Portfolio Variance

◇ The **optimal weight** is then solved as

$$\mathbf{w}^* = \frac{\boldsymbol{\Sigma}^{-1} \left((c\mu_p - b)\boldsymbol{\mu} + (a - b\mu_p)\mathbf{1} \right)}{ac - b^2}.$$

◇ The **portfolio variance** is a quadratic function of the **expected portfolio return** μ_p :

$$\begin{aligned} \sigma_P^2 &:= \mathbb{V}(r_p) = \mathbf{w}^{*\top} \boldsymbol{\Sigma} \mathbf{w}^* = \frac{c\mu_p^2 - 2b\mu_p + a}{ac - b^2} \\ &= \frac{c}{ac - b^2} \mu_p^2 - \frac{2b}{ac - b^2} \mu_p + \frac{a}{ac - b^2}. \end{aligned}$$

Global Minimum Variance Portfolio

- ◇ The **global minimum variance portfolio** is obtained by minimizing $\mathbb{V}(r_p)$ with respect to μ_p .

$$\frac{d\mathbb{V}(r_p)}{d\mu_p} = \frac{2c}{ac - b^2}\mu_p - \frac{2b}{ac - b^2}.$$

- ◇ The results of the **first-order conditions** are

$$\mu_{\star} = \frac{b}{c}, \quad \sigma_{\star}^2 = \frac{1}{c}, \quad \mathbf{w}_{\star} = \frac{\Sigma^{-1}\mathbf{1}}{c}.$$

Example

📌 For the two **assets**, compute the **inverse matrix**

$$\begin{aligned}\Sigma^{-1} &= \begin{bmatrix} 0.0625 & -0.01 \\ -0.01 & 0.16 \end{bmatrix}^{-1} \\ &= \frac{1}{0.0625 \times 0.16 - (-0.01) \times (-0.01)} \begin{bmatrix} 0.16 & 0.01 \\ 0.01 & 0.0625 \end{bmatrix} \\ &= \begin{bmatrix} 16.16 & 1.01 \\ 1.01 & 6.31 \end{bmatrix}\end{aligned}$$

Three Scalars

🔗 So the three **scalars** a , b , and c are

$$\begin{aligned} a &= \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \\ &= [0.05 \quad 0.08] \begin{bmatrix} 16.16 & 1.01 \\ 1.01 & 6.31 \end{bmatrix} \begin{bmatrix} 0.05 \\ 0.08 \end{bmatrix} \end{aligned}$$

$$= 0.088864;$$

$$\begin{aligned} b &= \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} \\ &= [0.05 \quad 0.08] \begin{bmatrix} 16.16 & 1.01 \\ 1.01 & 6.31 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

$$= 1.4441$$

$$\begin{aligned} c &= \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} \\ &= 24.49. \end{aligned}$$

Optimal Mean and Variance

📌 The **global minimum variance portfolio**

$$\mu_{\star} = \frac{b}{c} = 0.0590, \quad \sigma_{\star}^2 = \frac{1}{c} = 0.0408.$$

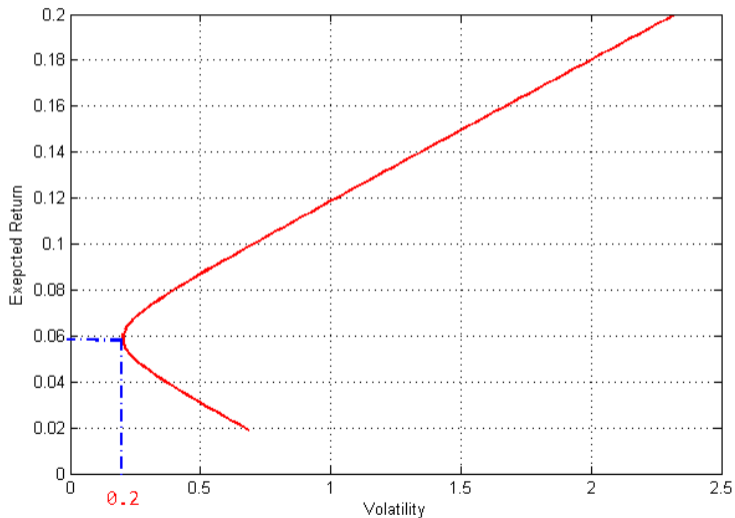
📌 So the minimum **volatility** is $\sigma_{\star} = 20.21\%$.

📌 The **weight vector** for w_{\star} for the global minimum variance portfolio is

$$w_{\star} = \frac{\Sigma^{-1}\mathbf{1}}{c} = \frac{1}{24.49} \begin{bmatrix} 16.16 & 1.01 \\ 1.01 & 6.31 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 70.10\% \\ 29.90\% \end{bmatrix}.$$

📌 So invest 70.1% of your money in Asset 1 and 29.9% in Asset 2 to earn 5.90% **expected return** with a **volatility** of 20.21%.

Efficient Frontier



Keywords

asset, 10, 11, 18

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