Lesson 6
Asset Pricing Models

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Broad Lesson Plan

Background

- Intuitively, a single market measure such as beta cannot capture all the information relevant to the price of a stock.
- Roll and Ross state that a security's long-run return is a function of changes in
  - Inflation
  - Industrial production
  - Risk premiums
  - The slope of the term structure of interest rates
- Generalization of **Capital Asset Pricing Model** (**CAPM**) is needed.

Portfolio Construction, Expected Return

- Consider a **portfolio** with w portion invested in an asset i of expected return $r_i := \mathbb{E}(r_{i,t})$ and $1-w$ portion invested in the **market portfolio** of **expected return** $r_m := \mathbb{E}\left(r_{m,t}\right)$.
- The return of this portfolio, denoted by $r_{w,t}$, is a weighted combination of $r_{i,t}$ and $r_{m,t}$.

$$r_{w,t} = wr_{i,t} + (1-w)r_{m,t} \tag{1}$$

- By the linear property of the expectation operator $\mathbb{E}\left(\,\cdot\,\right)$, the expected return of this portfolio is

$$r_w = wr_i + (1-w)r_m. \tag{2}$$

Lemma: Variance of $aX + bY$

- Let $\mu_X = \mathbb{E}(X)$, $\mu_Y = \mathbb{E}(Y)$, and $\mu_Z = \mathbb{E}(Z)$.
- Let $Z := aX + bY$.
- $\mu_Z = \mathbb{E}(Z) = a\,\mathbb{E}(X) + b\,\mathbb{E}(Y) = a\mu_X + b\mu_Y$
- The definition of variance of is

$$\begin{aligned}
\mathbb{V}\left(Z\right) &= \mathbb{E}\left((Z-\mu_Z)^2\right) = \mathbb{E}\left((aX - a\mu_X + bY - b\mu_Y)^2\right) \\
&= \mathbb{E}\left((aX - a\mu_X)^2 + (bY - b\mu_Y)^2 + 2(aX - a\mu_X)(bY - b\mu_Y)\right) \\
&= a^2\,\mathbb{E}\left((X-\mu_X)^2\right) + b^2\,\mathbb{E}\left((Y-\mu_Y)^2\right) \\
&\qquad + 2ab\,\mathbb{E}\left((X-\mu_X)(Y-\mu_Y)\right) \\
&= a^2\,\mathbb{V}(X) + b^2\,\mathbb{V}(Y) + 2ab\,\mathbb{C}(X,Y).
\end{aligned}$$

Variance of Portfolio's Return

- To (1), apply the variance operator $\mathbb{V}\left(\,\cdot\,\right)$. Using the lemma, we get

$$\mathbb{V}\left(r_{w,t}\right) = w^2\,\mathbb{V}\left(r_{i,t}\right) + (1-w)^2\,\mathbb{V}\left(r_{m,t}\right) + 2w(1-w)\,\mathbb{C}\left(r_{i,t}, r_{m,t}\right).$$

- For convenience, we denote
  - § $\sigma_w^2 := \mathbb{V}\left(r_{w,t}\right)$, $\sigma_i^2 := \mathbb{V}\left(r_{i,t}\right)$ and $\sigma_m^2 = \mathbb{V}\left(r_{m,t}\right)$
  - § The covariance $\sigma_{im} := \mathbb{C}\left(r_{i,t}, r_{m,t}\right)$.

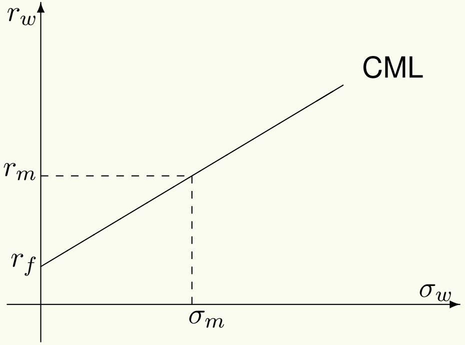
- With these notations, the variance $\mathbb{V}\left(r_{w,t}\right)$ simplifies to

$$\sigma_w^2 = w^2\sigma_i^2 + 2w(1-w)\sigma_{im} + (1-w)^2\sigma_m^2. \tag{3}$$

Slope of CML

- The **slope** of the **capital market line CML** is the **Sharpe ratio**. At $w = 0$, or for σ_m, we have

$$\frac{r_m - r_f}{\sigma_m} = \left.\frac{dr_w}{d\sigma_w}\right|_{w=0}.$$



Derivation of Slope

- It is tedious to compute $\dfrac{dr_w}{d\sigma_w}$ directly.
- Instead, we have, by chain rule, $\dfrac{dr_w}{d\sigma_w} = \dfrac{\frac{dr_w}{dw}}{\frac{d\sigma_w}{dw}}$.
- From (2), we obtain $\dfrac{dr_w}{dw} = r_i - r_m$.
- From (3), we obtain

$$2\sigma_w \frac{d\sigma_w}{dw} = 2w\sigma_i^2 + 2(1-2w)\sigma_{im}, -2(1-w)\sigma_m^2,$$

equivalently,

$$\frac{d\sigma_w}{dw} = \frac{w\sigma_i^2 + (1-2w)\sigma_{im} - (1-w)\sigma_m^2}{\sigma_w}.$$

Slope at $w = 0$

- Putting everything together,

$$\frac{dr_w}{d\sigma_w} = \frac{\frac{dr_w}{dw}}{\frac{d\sigma_w}{dw}} = \frac{r_i - r_m}{\frac{w\sigma_i^2 + (1-2w)\sigma_{im} - (1-w)\sigma_m^2}{\sigma_w}}.$$

- At $w = 0$, $\sigma_w = \sigma_m$, and given that the **slope** is the **Sharpe ratio**, we have

$$\frac{r_m - r_f}{\sigma_m} = \frac{r_i - r_m}{\left(\frac{\sigma_{im} - \sigma_m^2}{\sigma_m}\right)} \quad \Longrightarrow \quad r_m - r_f = \frac{r_i - r_m}{\left(\frac{\sigma_{im} - \sigma_m^2}{\sigma_m^2}\right)} = \frac{r_i - r_m}{\left(\frac{\sigma_{im}}{\sigma_m^2} - 1\right)}.$$

Slope at $w = 0$ (Cont'd)

- For any asset i that is not a **market portfolio**, $\frac{\sigma_{im}}{\sigma_m^2} - 1 \neq 0$. So we multiple it to both sides to obtain

$$(r_m - r_f)\left(\frac{\sigma_{im}}{\sigma_m^2} - 1\right) = r_i - r_m,$$

$$\Longrightarrow \qquad \frac{\sigma_{im}}{\sigma_m^2}(r_m - r_f) - (r_m - r_f) = r_i - r_m.$$

- Knowing that $\frac{\sigma_{im}}{\sigma_m^2} = \beta_i$, we write,

$$\beta_i(r_m - r_f) = (r_m - r_f) + r_i - r_m = r_i - r_f$$

- Hence **CAPM** ensues:

$$\boxed{r_i - r_f = \beta_i(r_m - r_f)}\,.$$

A Lemma

Lemma

Let $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_K$ and $\boldsymbol{z}$ be N-dimensional column vectors. Suppose another N-dimensional column vector, $\boldsymbol{w}$, is orthogonal to all these $K+1$ vectors. In other words,

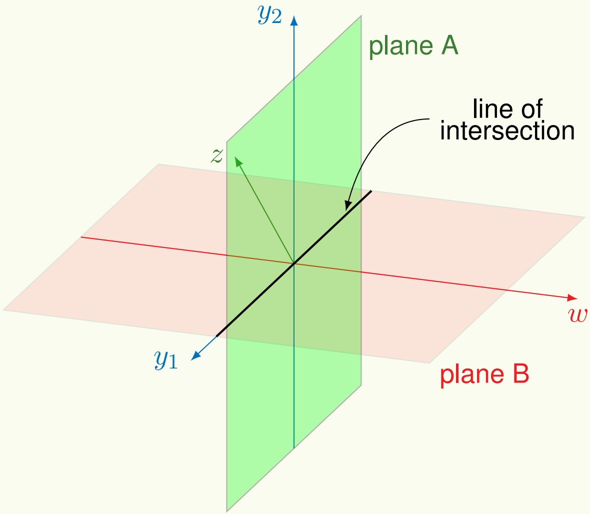
$$\boldsymbol{w}^\intercal \boldsymbol{z} = 0; \quad \boldsymbol{w}^\intercal \boldsymbol{y}_i = 0, \qquad i = 1, 2, \ldots, K.$$

Then, $\boldsymbol{z}$ is a **linear combination** of $\boldsymbol{y}_i$, $i = 1, 2, \ldots, K$:

$$\boldsymbol{z} = \sum_{i=1}^{K} \lambda_i \boldsymbol{y}_i,$$

where $\lambda_i \neq 0$ for all i.

Intuitive Illustration



- Plane A and Plane B are perpendicular.
- Any vector on Plane A is perpendicular to a vector $\boldsymbol{w}$ on Plane B.
- Clearly, vector $\boldsymbol{z}$ can be expressed as a **linear combination** of $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ on Plane A.

Proof ot Lemma

- Since $\boldsymbol{w}^\intercal \boldsymbol{y}_i = 0,\ i = 1, 2, \ldots, K$, for non-zero λ_i,

$$\sum_{i=1}^{K} \left(-\lambda_i \boldsymbol{w}^\intercal \boldsymbol{y}_i \right) = 0.$$

- Adding $\boldsymbol{w}^\intercal \boldsymbol{z} = 0$ to the above, we have $\boldsymbol{w}^\intercal \boldsymbol{z} - \sum_{i=1}^{K} \lambda_i \boldsymbol{w}^\intercal \boldsymbol{y}_i = 0.$
- Factoring out $\boldsymbol{w}^\intercal$, we obtain $\boldsymbol{w}^\intercal \left(\boldsymbol{z} - \sum_{i=1}^{K} \lambda_i \boldsymbol{y}_i \right) = 0.$
- Since $\boldsymbol{w}$ is any arbitrary vector, it must be that $\boldsymbol{z} = \sum_{i=1}^{K} \lambda_i \boldsymbol{y}_i.$ □

APT Assumptions

- Assume that there are M risk factors F_k, $k = 1, 2, \ldots, M$.
- Without loss of generality, the means of these factors are all zero.
- Suppose there are N assets, each with return r_i, and

$$r_i = a_i + \sum_{k=1}^{M} b_{ik} F_k + e_i,$$

  where a_i and b_{ik} for $k = 1, 2, \ldots, M$ are constants specific to the i^{th} asset.
- Motivated by the paradigm of **ordinary least squares**, "**explanatory variables**" F_k should not have correlation with "noise" e_t.

Portfolio Construction

- Suppose we can form a large portfolio of n_1 assets with equal weight $x_i = \dfrac{1}{n_1}$ such that $\displaystyle\sum_{i=1}^{n_1} x_i = 1$.
- By the **law of large numbers**, $\displaystyle\sum_{i=1}^{n_1} x_i e_i = \frac{1}{n_1}\sum_{i=1}^{n_1} e_i \approx 0$.
- Then this portfolio's **expected return** R_1 is $R_1 = A_1 + \displaystyle\sum_{k=1}^{M} B_{1k}F_k$, where $A_1 = \displaystyle\sum_{i=1}^{n_1} x_i a_i$ and $B_{1k} = \displaystyle\sum_{i=1}^{n_1} x_i b_{ik}$ for all $k = 1, 2, \ldots, M$.

Large Number of Portfolios

- Likewise, we form a second large portfolio of n_2 assets (distinct from those in the first portfolio) with the same method. This portfolio's **expected return** is

$$R_2 = A_2 + \sum_{k=1}^{M} B_{2k} F_k,$$

where $A_2 = \sum_{i=1}^{n_2} x_i a_i$, and $B_{2k} = \sum_{i=1}^{n_2} x_i b_{ik}$ for all k

- Suppose we can form Q such almost **equal-weight portfolios**. Since each asset is used only once in portfolios construction, it must be that $\sum_{i=1}^{Q} n_i = N$.

A Portfolio of Portfolios

- For Q portfolios, we have $\boldsymbol{R} := \begin{bmatrix} R_1 & R_2 & \cdots & R_Q \end{bmatrix}^\top$.

- Likewise, we collect the coefficients as vector $\boldsymbol{A}$ and matrix $\boldsymbol{B}$:

$$\boldsymbol{A} := \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_Q \end{bmatrix} \text{ and } \boldsymbol{B} := \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ B_{Q1} & B_{Q2} & \cdots & B_{QM} \end{bmatrix}.$$

- We want to construct a **self-financing** and **factor-neutral** portfolio from these Q portfolios.

- The weight vector of this portfolio of portfolios is denoted by $\boldsymbol{w} := \begin{bmatrix} w_1 & w_2 & \cdots & w_Q \end{bmatrix}^\top$.

Factor Neutrality

- Let $\mathbf{1}$ be a vector with all elements being 1.
- Self-financing means **zero cash outlay**:

$$\boldsymbol{w}^\intercal \mathbf{1} = \sum_{i=1}^{Q} w_i = 0.$$

- Factor neutrality means **orthogonality**:

$$\boldsymbol{w}^\intercal \boldsymbol{B}_k = 0, \qquad k = 1, 2, \ldots, M,$$

where $\boldsymbol{B}_k = \begin{bmatrix} B_{1k} & B_{2k} & \cdots & B_{Qk} \end{bmatrix}^\intercal$ is the k^{th} column of matrix $\boldsymbol{B}$.

Expected Return on a Portfolio of Portfolios

- The portfolio of portfolios has the return given by

$$
\begin{aligned}
\boldsymbol{w}^\intercal \boldsymbol{R} &= \sum_{i=1}^{Q} w_i R_i = \sum_{i=1}^{Q} w_i \left(A_i + \sum_{j=1}^{M} B_{ij} F_j \right) \\
&= \sum_{i=1}^{Q} w_i A_i + \sum_{i=1}^{Q} \sum_{j=1}^{M} w_i B_{ij} F_j \\
&= \boldsymbol{w}^\intercal \boldsymbol{A} + \sum_{j=1}^{M} \sum_{i=1}^{Q} (w_i B_{ij}) F_j \\
&= \boldsymbol{w}^\intercal \boldsymbol{A} + \sum_{j=1}^{M} (\boldsymbol{w}^\intercal \boldsymbol{B}_j) F_j \\
&= \boldsymbol{w}^\intercal \boldsymbol{A}.
\end{aligned}
$$

Input from $\mathbb{Q}$uantitative Finance

- But since there is no cash outlay, the **no-arbitrage principle** requires that this portfolio return must be zero.

$$\boldsymbol{w}^\intercal \boldsymbol{R} = 0$$

- It follows that vector $\boldsymbol{w}^\intercal$ is **orthogonal** to vectors $\boldsymbol{R}$, $\mathbf{1}$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, and so on to $\boldsymbol{B}_M$.

- By the lemma, for $\lambda_j \neq 0, j = 0, 1, 2, \ldots, M$, we obtain a linear model of multiple (M) factors F_j:

$$\boldsymbol{R} = \lambda_0 \mathbf{1} + \sum_{j=1}^{M} \lambda_j \boldsymbol{B}_j. \quad (4)$$

Special Cases

- If we set all $\lambda_j = 0\ j \neq 0$, then there is no risk and $\lambda_0 = r_f$, the **risk-free rate**.
- $M = 1$, i.e. one-factor model. Suppose this factor is the **market factor**, i.e., $\lambda_1 = \mathbb{E}(R_m - r_f)$, where R_m is the expected return on **market portfolio**.
- Then (4) becomes

$$R_{Q\times 1} = r_f 1_{Q\times 1} + \mathbb{E}(R_m - r_f) B_{Q\times 1},$$

  which is **CAPM**.
- Thus, **APT** (4) is a general result. The values $\lambda_j,\ j \neq 0$ are the **price of risk**, or **risk premium** associated with the different factors F_j, $j = 1, 2, \ldots, M$.

Factors in French's Data Library

Rm − Rf	market risk premium (market factor)
SMB	size factor
HML	value factor
RMW	profitability factor
CMA	investment factor
UMD	momentum factor
STR	short-term reversal factor
LTR	long-term reversal factor

Remarks

- Risk factors are expressed as returns in percent at the stated frequency (daily, weekly, **monthly**, annually)
  - Expected return is proportional to risk!
  - The proportional constant is the “beta”.
  - Beta is a measure of the sensitivity to risk factor.
- Factor calculations are based on **CRSP US Stocks & Index Databases** and **CRSP 1925 Historical Indexes Guide**.
  - firms listed on NYSE, AMEX, and NASDAQ for U.S. factors
- Value factor may be redundant (Fama & French (2015)).

Questions

- What are the assumptions of arbitrage pricing theory?
- Liquidity risk of trading is a genuine concern. Is it a non-redundant factor?
- Is there an algorithm to construct a portfolio with no exposure whatsoever to risk?
- Answer: One of the ways is to use the **algorithm of principal component analysis**.

Takeaways

1. Arbitrage pricing theory is a one-period model that generalizes one-period CAPM.
2. It is based on the principle of no risk-free arbitrary opportunity.
   - If the portfolio (of portfolios) has no exposure to any risk factor, it will not “earn” its risk premium.
   - If the portfolio is constructed with zero cost and zero risk exposure, the expected return will have to be zero.
3. Duality of risk and expected return

$$\text{Risk factor} \longleftrightarrow \text{risk Premium}$$

Additional Reading

1. **A Practitioner's Guide to Arbitrage Pricing Theory**

2. **A Five-Factor Asset Pricing Model**, Fama and French, Journal of Financial Economics, Volume 116, 1-22, 2015

3. **Taming the Factor Zoo: A Test of New Factors**, Feng, Giglio, and Xiu, Journal of Finance, Volume 75, 1327-1370, 2017

Keywords