Lesson 3
Prices and Returns

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Sampling

Definition 1.1 (Time Series).

In **time series analysis**, **regular sampling** is a data collection scheme that is based on the **clock time**. On the other hand, **irregular sampling** is based on the **business time**, which is the arrival time of an event that gives rise to a set of numbers to be recorded as a sample.

- For **regular sampling**, t is the clock time *by* which the last transaction of the trading day or trading session takes place.
- For **irregular sampling**, t is the time *at* which a trade occurs. Each trade is identified by a serial number t, which indicates its chronological order in the time series.

Time Series

Definition 1.2 (Time Series).

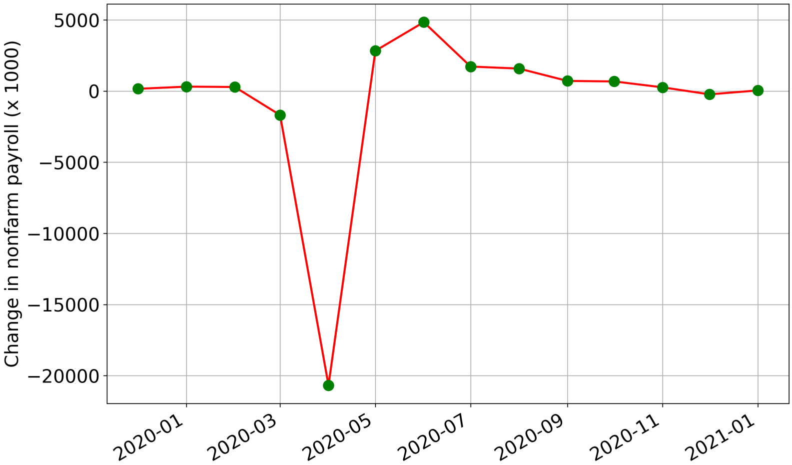
We define **time series** as a chronologically arranged sequence of quantities sampled by applying a sampling scheme consistently. The time series of prices is denoted by P_t, for $t = 1, 2, \ldots, T$, where T indicates the last or latest observed value in the sample.

- The same sampling scheme must be applied consistently. We *must* stick to the same sampling scheme throughout the process of recording a time series.
- We have implicitly assumed that the time series is discrete with respect to time t, i.e., the **time interval** between any pair of consecutive points in the time series is a finite value.

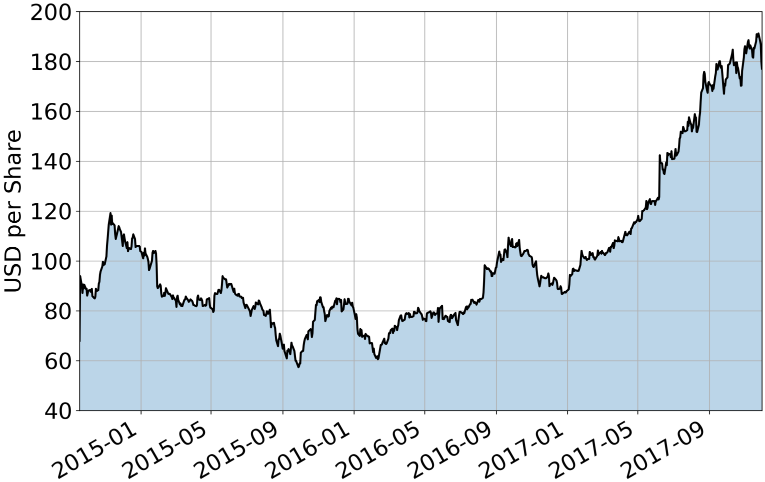
Examples of Economic and Financial Time series

Event	Quantity	Time	Remarks
Transaction	Price	Clock	Usually last traded price, daily
Transaction	Intraday price	Clock	Usually 5-minute interval
Transaction	Tick-by-tick price	Business	High frequency
Earnings Announcement	Earnings per share	Business	Scheduled
Company Distribution	Dividend per share	Business	Ex date
US Employment Situation	Nonfarm payrolls	Clock	Every first Friday of the month
US ISM Manufacturing Index	Index level	Clock	First business day of the month

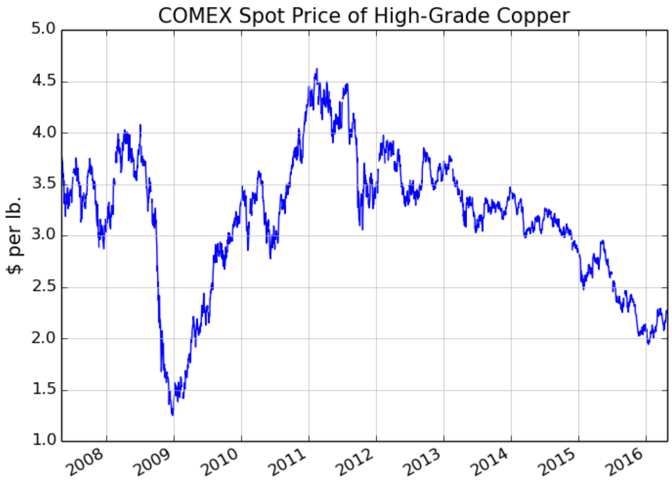
Monthly Time Series of Changes in Nonfarm Payrolls



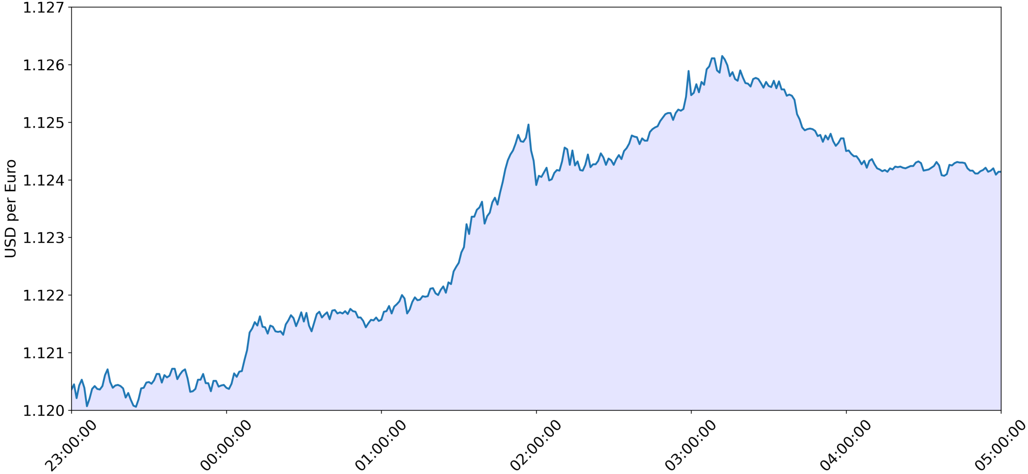
Stock Prices of Alibaba Group Holding Ltd Since IPO



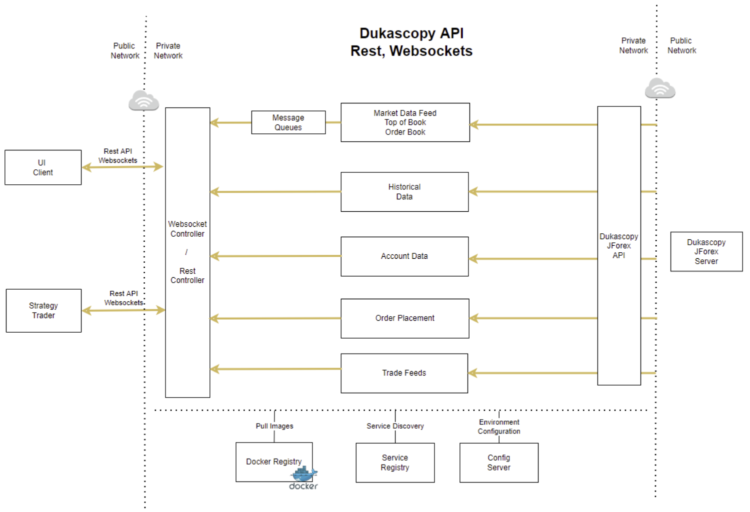
Spot Copper Prices



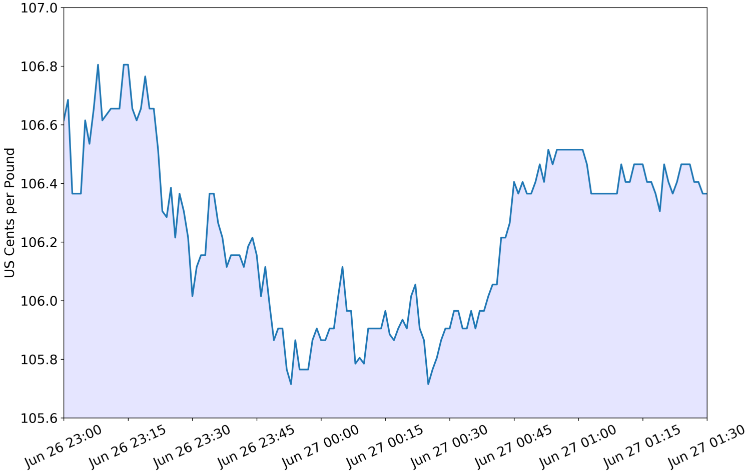
Foreign Exchange (FX)



FinTech


Dukascopy API
Rest, Websockets
Public Network
Private Network
Private Network
Public Network
UI Client
Rest API Websockets
Strategy Trader
Rest API Websockets
Websocket Controller / Rest Controller
Message Queues
Market Data Feed Top of Book Order Book
Historical Data
Account Data
Order Placement
Trade Feeds
Dukascopy JForex API
Dukascopy JForex Server
Pull Images
Service Discovery
Environment Configuration
Docker Registry
docker
Service Registry
Config Server

Coffee



US Cents per Pound
107.0
106.8
106.6
106.4
106.2
106.0
105.8
105.6
Jun 26 23:00
Jun 26 23:15
Jun 26 23:30
Jun 26 23:45
Jun 27 00:00
Jun 27 00:15
Jun 27 00:30
Jun 27 00:45
Jun 27 01:00
Jun 27 01:15
Jun 27 01:30

US Dollar Index

- Started in March 1973 with a base of 100, the **US Dollar Index** (**USDX**) is an indicator of the value of US dollar against a basket of six currencies.
- They are Euro (EUR), Japanese yen (JPY), British pound (GBP), Canadian dollar (CAD), Swedish krona (SEK), and Swiss franc (CHF).
- Currently, the dollar index is maintained and published by ICE (Intercontinental Exchange, Inc.). The US Dollar Index is calculated with the following formula:

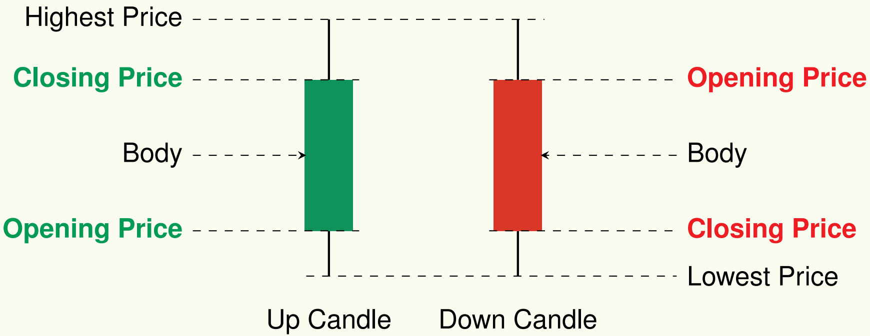
$$\begin{aligned} \text{USDX} =& 50.14348112 \times \text{EURUSD}^{-0.576} \times \text{USDJPY}^{0.136} \times \text{GBPUSD}^{-0.119} \\ & \times \text{USDCAD}^{0.091} \times \text{USDSEK}^{0.042} \times \text{USDCHF}^{0.036}. \end{aligned}$$

Dollar Index toward the End of the First Week of February 2021

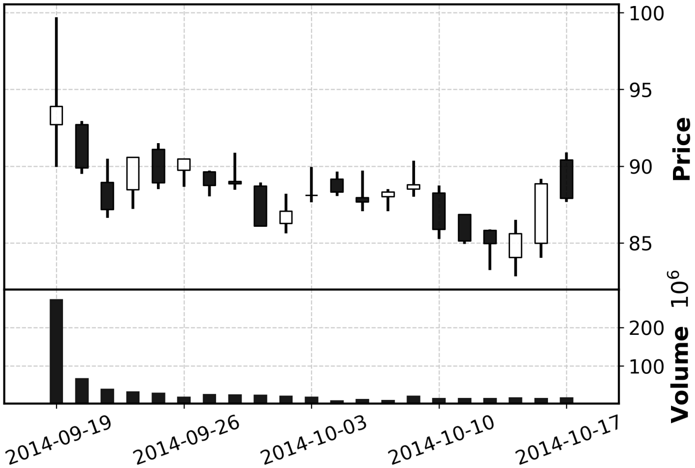
Date	Time	USDX
06.02.2021	06:58:00.087 GMT+0900	90.9970
06.02.2021	06:58:00.352 GMT+0900	90.9955
06.02.2021	06:58:13.466 GMT+0900	90.9950
06.02.2021	06:58:21.783 GMT+0900	90.9931
06.02.2021	06:58:29.437 GMT+0900	90.9965
06.02.2021	06:58:40.560 GMT+0900	90.9961
06.02.2021	06:58:54.909 GMT+0900	90.9927
06.02.2021	06:58:58.466 GMT+0900	90.9924
06.02.2021	06:59:02.393 GMT+0900	90.9929
06.02.2021	06:59:18.323 GMT+0900	90.9920
06.02.2021	06:59:23.846 GMT+0900	90.9944
06.02.2021	06:59:30.222 GMT+0900	90.9963
06.02.2021	06:59:35.114 GMT+0900	90.9955
06.02.2021	06:59:36.907 GMT+0900	90.9939
06.02.2021	06:59:45.278 GMT+0900	90.9885
06.02.2021	06:59:55.583 GMT+0900	90.9868

Data visualization by the Japanese Candlesticks

Up candle and the Down candle



Alibaba's Open-High-Low-Close for the First 4 Weeks



Profit & Loss and Price Change

- Buy first and sell later; (short) sell first then buy back

Profit and Loss = Selling price − Buying price.

Definition 2.1 (Price Change).

The **price change** at time t of a price series P_t, for $t = 0, 1, 2, \ldots, T$, is a **time series** given by the **price difference** of a pair of adjacent prices.

$$\Delta P_t := P_t - P_{t-1},$$

for $t = 1, 2, \ldots, T$, where T is the last observation time of the sampled time series. The **IPO price** of the stock is denoted by P_0.

Simple Return

Definition 2.2.

The **simple return**, denoted by R_t, of a price series P_t for $t = 0, 1, 2, \ldots, T$ is a time series given by the price differences of any pair of adjacent prices divided by P_{t-1}.

$$R_t := \frac{P_t - P_{t-1}}{P_{t-1}},$$

for $t = 1, 2, \ldots, T$, where T is the last observation time of the sampled time series.

- Why the price change $\Delta P_t = P_t - P_{t-1}$ is divided by P_{t-1} and not P_t?
- The simple return can be re-expressed as

$$R_t = \frac{P_t}{P_{t-1}} - 1. \tag{1}$$

Payoff Ratio

Definition 2.3.

The **payoff ratio** is defined as $\frac{P_t}{P_{t-1}}$. It indicates, in terms of per dollar capital, the amount an investor will either win or lose in the investment.

- If $P_t = 0$, from (1), them $R_t = -1$.
- But the stock $P_t \geqq 0$. Therefore
$$R_t \geqq -1.$$
- For some applications, the lower bound could be a hindrance.

Payoff Ratio and Log Return

- To overcome this problem, we start with the **payoff ratio** $\frac{P_t}{P_{t-1}}$, which is never negative as $P_t \geqq 0$.
- We then consider the natural logarithm of the payoff ratio.

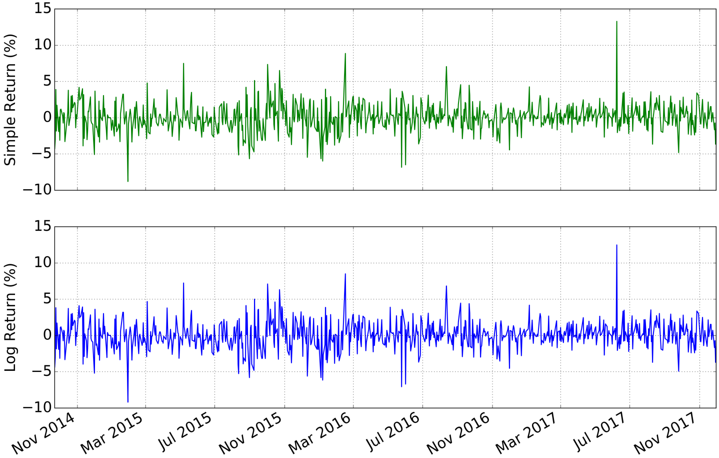
Definition 2.4.

The **log return**, denoted by r_t, of a price series P_t for $t = 0, 1, 2, \ldots, T$ is a time series given by the differences of adjacent log prices. That is

$$r_t := \ln\left(\frac{P_t}{P_{t-1}}\right) = \ln P_t - \ln P_{t-1},$$

for $t = 1, 2, \ldots, T$, where T is the last observation time of the sampled time series.

Simple Return and Log Return



Relation between Simple Return and Log Return

- Log return as a function of simple return

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right) = \ln(1 + R_t). \tag{2}$$

- Usually $|R_t| \ll 1$, i.e., the absolute value of the simple return is much smaller than 1.
- We perform **Taylor's expansion** of $\ln(1 + R_t)$ and obtain, up to the second order,

$$r_t = \ln(1 + R_t) = R_t - \frac{1}{2}R_t^2 + O\big(R_t^3\big),$$

where $O\big(R_t^3\big)$ denotes all the remaining terms of third and higher orders.

- With the simple return being small, even the second-order term can be ignored, resulting in $r_t \approx R_t$. In fact, $r_t \leqq R_t$.

Modeling Randomness in Prices and Returns

Definition 2.5.

Let the payoff ratio M_t be a strictly positive random variable at time t. For emphasis, we write $M_t > 0$ for all t. Consider a **time series** of M_t. A model of asset prices P_t is as follows:

$$P_t = P_{t-1}M_t.$$

Equivalently, we have a model of random logarithmic asset prices:

$$\ln P_t = \ln P_{t-1} + \ln M_t,$$

for $t = 1, 2, \ldots, T$.

- Log return is noise.

$$r_t = \ln P_t - \ln P_{t-1} = \ln M_t \qquad \Longrightarrow \qquad r_t = \ln M_t.$$

Telescopic Multiplication and q-daily log return

- An interesting property of the **payoff ratio** in the context of multi-period return is called the **rule of telescopic multiplication**:

$$\frac{P_t}{P_{t-q}} = \frac{P_t}{P_{t-1}} \times \frac{P_{t-1}}{P_{t-2}} \times \frac{P_{t-2}}{P_{t-3}} \times \cdots \times \frac{P_{t-q+2}}{P_{t-q+1}} \times \frac{P_{t-q+1}}{P_{t-q}} \tag{3}$$

- When we apply the natural logarithm on both sides of, (3), we obtain

$$r_{q,t} = r_t + r_{t-1} + r_{t-2} + \cdots + r_{t-q+2} + r_{t-q+1},$$

where $r_{q,t}$ is a notation for **q-daily log return**:

$$r_{q,t} := \ln\left(\frac{P_t}{P_{t-q}}\right) = \ln P_t - \ln P_{t-q}.$$

- Therefore, q-daily log return is a sum of q daily log returns.

Arithmetic Average Log Return

- The exponential function is the reverse function of logarithm, i.e., $\exp\big(\ln(x)\big) = x$. Thus, another property of **multi-period log return** is that

$$\exp\left(r_{q,t}\right) = \frac{P_t}{P_{t-q}}.$$

- By definition,

$$P_t = P_{t-q}\exp\left(r_{q,t}\right) = P_{t-q}\exp\Big(r_t + r_{t-1} + r_{t-2} + \cdots + r_{t-q+2} + r_{t-q+1}\Big).$$

- From time $t-q+1$ to t, there are q periods. We write the **arithmetic average log return** as

$$\bar{r}_t := \frac{1}{q}\Big(r_t + r_{t-1} + r_{t-2} + \cdots + r_{t-q+2} + r_{t-q+1}\Big).$$

- It follows that $P_t = P_{t-q}\exp\left(q\bar{r}_t\right) \qquad \Longrightarrow \qquad r_{q,t} = q\bar{r}_t.$

Geometric Average Return

- Now, from the institutional investment perspective, one of the greatest concerns of any fund manager of a portfolio is the **asset under management** (**AUM**). A more relevant return to fund managers is the notion of **geometric average** over a number of years.

Definition 3.1.

The **geometric average return**, denoted by g_t, is defined primarily for calculating the average rate of return per period on investments that are compounded over multiple periods. It is defined with respect to the payoff ratio:

$$g_t := \left(\frac{P_t}{P_{t-q}}\right)^{\frac{1}{q}} - 1. \tag{4}$$

Geometric Return and Simple Return

- By the rule of telescopic multiplication (3), we can rewrite g_t as

$$1 + g_t = \left(\frac{P_t}{P_{t-1}} \times \frac{P_{t-1}}{P_{t-2}} \times \frac{P_{t-2}}{P_{t-3}} \times \cdots \times \frac{P_{t-q+2}}{P_{t-q+1}} \times \frac{P_{t-q+1}}{P_{t-q}} \right)^{\frac{1}{q}}.$$

- Each period's **payoff ratio** is related to the **simple return**, i.e., $\dfrac{P_{t-i}}{P_{t-i-1}} = 1 + R_{t-i}$ for $i = 0, 1, 2, \ldots, q$. Consequently,

$$(1 + g_t)^q = (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-q+1}). \tag{5}$$

- Intuitively, what we find is that if we hold the investment throughout the q periods, every dollar invested will become $(1 + g_t)^q$ dollars.

Geometric Average Return is Larger than Average Log Returns

Proposition 1

The **geometric average return** g_t is always larger than the arithmetic average of the log returns. That is, it must be that

$$g_t \geqq \overline{r}_t.$$

Proof.

- To extract g_t, we take logarithm on both sides of (5) to yield

$$q \ln(1+g_t) = \ln(1+R_t) + \ln(1+R_{t-1}) + \cdots + \ln(1+R_{t-q+1}).$$

- Noting the relationship between log return and simple return, (2), we have

$$\ln(1+g_t) = \frac{r_t + r_{t-1} + \cdots + r_{t-q+1}}{q} = \overline{r}_t. \tag{6}$$

Proof (cont'd)

Proof.

- It leads to $1 + g_t = \mathrm{e}^{\overline{r}_t}$.
- Since $|\overline{r}_t| \ll 1$, by Maclaurin's expansion of exponential function, we obtain

$$g_t = \mathrm{e}^{\overline{r}_t} - 1 = \overline{r}_t + \frac{1}{2}\overline{r}_t^2 + O\left(\overline{r}_t^3\right) \approx \overline{r}_t + \frac{1}{2}\overline{r}_t^2.$$

- Thus, we see that the geometric average return g_t is greater than the arithmetic average of the log returns by an amount of approximately $\frac{1}{2}\overline{r}_t^2$.

□

- The geometric return g_t in the case of one period is equal to the simple return R_t, and we have $g_t = R_t \geqq r_t$. They are equal in the trivial case when both returns are 0.

Calculating Client's Return of Investment

- In the investment industry, investors at times will invest more by infusion of fresh money.
- Conversely, at times, investors will invest less by withdrawing money from their investment accounts.
- How should we, as the fund manager, compute some sort of average return for the investors?
- The answer to this question is a two- the calculation of simple returns followed by computing the geometric average return.
- The resulting average is referred to as the **time-weighted return**.

Example 4.8

- An institutional investor, Rotsevni, invests $1 million into a fund on December 31. Rotsevni is the only client.
- Ten months later on October 31 the following year, through tactical and strategic investments, the value of the portfolio becomes $1.2 million.
- On that day, Rotsevni invests $0.8 million more on October 31, bringing the **asset under management** to $2 million.
- By the end of the year, the portfolio value becomes $1.9 million because a particular blue-chip stock in the portfolio is in trouble, and its share price plunges.
- The fund need to report to your client, and the obvious question is, “What is the return?”

Example (cont'd)

- For the first 10 months, the simple return is

$$\frac{1.2 - 1.0}{1.0} = 20\%.$$

- For the next two months, the simple return is

$$\frac{1.9 - 2}{2} = -5\%.$$

- Having computed the simple returns, the fund manager then proceeds to compute the **annual return** for Rotsevni by the **geometric average**

$$(1 + 0.20) \cdot (1 - 0.05) = 1.14.$$

- Therefore, the time-weighted return is $(1.14 - 1) = 14\%$.

Example 4.9

- Suppose a fund is investing on behalf of its only client as in Example 4.8.
- Again, the portfolio grows by 20% over the first 10 months.
- Instead of injecting more fund, Rotsevni withdraws \$0.2 million, bringing the **AUM** to \$1.2 million − \$0.2 million = \$1 million, as of October 31.
- Likewise, the portfolio takes a knock and its value becomes \$0.95 million by the end of the year.
- The fund need to report to your client, and the obvious question is, “What is the return?”

Example 4.9 (cont'd)

- For the first 10 months, the return is 20% as before.
- For the last two months of the year, the simple return is

$$\frac{0.95 - 1}{1} = -5\%.$$

- The geometric average return is again $(1 + 0.2) \cdot (1 - 0.05) - 1 = 14\%$.

GIC

- In 1981, Mr Goh Keng Swee, then chairman of the Monetary Authority of Singapore, saw the danger of Singapore's growing foreign reserves in the midst of heightened inflation risk.

- Being also the first Deputy Prime Minister, he rolled out an initiative to set up the Government of Singapore Investment Corporation Pte Ltd (GIC), with the mandate to invest Singapore's foreign reserves, so as to earn reasonable returns within acceptable risk limits over the long term.

- GIC is one of the so-called sovereign wealth funds in the world. As the name suggests, a **sovereign wealth fund** is a state-owned investment vehicle to manage national budget surpluses, accumulated over the years due to favorable macroeconomic, trade, and fiscal positions, coupled with long-term budget planning under spending restraint.

Nominal Return and Real Return

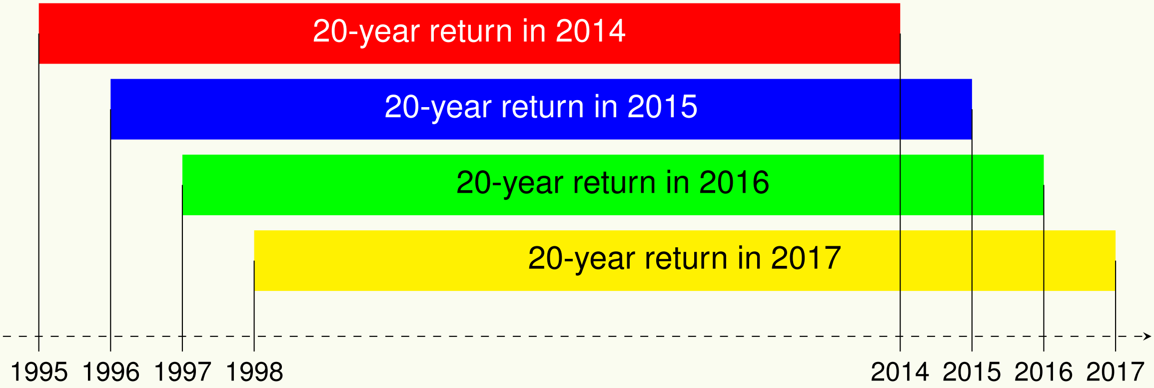
Definition 5.1.

The **nominal return** is defined as the return that does not take inflation effects into account. The **real return**, on the other hand, is the return adjusted for changes in the price levels due to **inflation**.

- GIC's 20-year annualized **nominal return** in percent calculated based on US dollars and GIC's 20-year annualized **real return**

Year	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017
Nominal	9.5	9.5	8.7	9.5	9.7	8.5	7.9	7.7	5.7	7.1	7.2	6.8	6.5	6.5	6.1	5.7	5.7
Real	5.8	5.8	4.5	5.0	4.9	4.9	4.8	4.5	2.6	3.8	3.9	3.9	4.0	4.1	4.9	4.0	3.7

Illustration of Rolling 20-Year Return



Portfolio Value

- Next, consider the 20-year nominal geometric average return of 5.7% in the 2017 annual report.
- Conceptually, it corresponds to the return obtained from comparing the GIC portfolio value of March 1997 with that of March 2017*.
- Applying (4), we obtain

$$\left(\frac{P_{2017}}{P_{1997}}\right)^{\frac{1}{20}} - 1 = 5.7\%.$$

- To get a more intuitive picture, we rewrite this equation as

$$P_{2017} = P_{1997}(1 + 0.057)^{20}.$$

Comparison with Inflation

- To make the concept of geometric average return more concrete, suppose we had US $100 in 1997 and suppose we could invest our money in the exact way GIC invests.
- Our $100 would become $\$100 \times (1.057)^{20} = \$303.04$ in 2017.
- Given that the corresponding 20-year real return is 3.7%, for which inflation has been adjusted, our purchasing power in 2017 would be

$$\$100 \times (1.037)^{20} = \$206.81,$$

which is twice more than we could afford to buy 20 years ago.

More or Less?

Proposition 2

If the 20-year geometric return this year is smaller (bigger) than that of the last year, then the simple return over the past one year is less (more) than the simple return 20 years ago.

Proof.

- Let g_t and g_{t-1} be the m-year geometric return for the reports published in year t and year $t-1$, respectively.
- Suppose R_t is the 1-year simple return—the return made over the past year, i.e., from $t-1$ to t.
- Also, suppose R_{t-m} is the 1-year simple return obtained from year $t-m-1$ to $t-m$. □

Proof (cont'd)

Proof.

- By the definition of rolling window by a year,

$$(1 + R_{t-m})G = \left(1 + g_{t-1}\right)^m$$

and

$$G(1 + R_t) = \left(1 + g_t\right)^m,$$

where G is the product $\displaystyle\prod_{i=t-m+1}^{t-1} (1 + g_i)$, which is common to both m-year geometric returns.

- Note that G is necessarily positive because the simple return is strictly bounded from below by -1.

Proof (cont'd)

Proof.

- Now, the difference of these two expressions is

$$(1+R_t)-(1+R_{t-m}) = \frac{(1+g_t)^m-(1+g_{t-1})^m}{G}.$$

- Thus we see that the sign of $R_t - R_{t-m}$ is dependent on the numerator $(1+g_t)^m-(1+g_{t-1})^m$.
- Being a monotonic power function, the sign of the simple return difference will be positive if on the right-hand side, $g_t > g_{t-1}$.
- Let $m = 20$ and the proof is complete.

□

Ex Date, Record Date, Payment Date

- **Ex date** is the cutoff date before which existing and new shareholders are entitled to receive the upcoming dividend payments.
- **Record date** is the date at which the book containing the particulars of each shareholder such as the number of shares owned, mailing address, etc are updated and closed.
- **Payment date** is the earliest date on which you will receive your dividend.
- See next slide for a portion of the dividend history of Coca-Cola. Data source: **Nasdaq**.

Risk-Free Arbitrage

- By **no risk-free arbitrage**, the stock price should drop by an amount equal to the dividend per share D_t on ex date.
- Suppose the stock price does not change from $t-1$ to t. Investors will always buy the stock at day $t-1$ and sell it on ex dividend day t, and they will receive the dividend without risk.
- Therefore, holding all the market conditions constant, the share price on ex date t will have to drop by D_t.
- So it is ex-date that counts for receiving D_t.

Declaration Date	Type	Amount	Ex Date	Record Date	Payment Date
2017-10-19	Cash	$0.37	2017-11-30	2017-12-01	2017-12-15
2017-07-20	Cash	$0.37	2017-09-14	2017-09-15	2017-10-02
2017-04-27	Cash	$0.37	2017-06-13	2017-06-15	2017-07-03
2017-02-16	Cash	$0.37	2017-03-13	2017-03-15	2017-04-03
2016-10-20	Cash	$0.35	2016-11-29	2016-12-01	2016-12-15
2016-07-21	Cash	$0.35	2016-09-13	2016-09-15	2016-10-03
2016-04-28	Cash	$0.35	2016-06-13	2016-06-15	2016-07-01
2016-02-18	Cash	$0.35	2016-03-11	2016-03-15	2016-04-01
2015-10-15	Cash	$0.33	2015-11-27	2015-12-01	2015-12-15
2015-07-16	Cash	$0.33	2015-09-11	2015-09-15	2015-10-01
2015-04-30	Cash	$0.33	2015-06-11	2015-06-15	2015-07-01
2015-02-19	Cash	$0.33	2015-03-12	2015-03-16	2015-04-01
2014-10-16	Cash	$0.305	2014-11-26	2014-12-01	2014-12-15
2014-07-15	Cash	$0.305	2014-09-11	2014-09-15	2014-10-01
2014-04-24	Cash	$0.305	2014-06-12	2014-06-16	2014-07-01
2014-02-20	Cash	$0.305	2014-03-12	2014-03-14	2014-04-01
2013-10-17	Cash	$0.28	2013-11-27	2013-12-02	2013-12-16
2013-07-18	Cash	$0.28	2013-09-12	2013-09-16	2013-10-01

Total Return

Definition 6.1.

The **total return**, denoted by $\breve{R}$, is the return that recognizes dividend D_t per share as the cash flow receipt in the P&L computation, resulting in

$$\breve{R}_t := \frac{P_t + D_t - P_{t-1}}{P_{t-1}}. \tag{7}$$

Definition 6.2.

The ratio of dividend D_t to stock price P_t is called the **dividend yield**.

Inferring the Dividend Yield

Proposition 3

If the total return and the simple return are given, then the dividend yield at time t can be inferred by the following formula:

$$\frac{D_t}{P_t} = \frac{\breve{R}_t - R_t}{1 + R_t}. \tag{8}$$

Proof.

- First, we express the total return (7) as $\breve{R}_t = \dfrac{D_t}{P_{t-1}} + \dfrac{P_t - P_{t-1}}{P_{t-1}} = \dfrac{D_t}{P_{t-1}} + R_t$.
- Shifting R_t to the left-hand side, and multiplying the dividend yield by $1 = \dfrac{P_t}{P_t}$, we obtain, after swapping the denominators, $\breve{R}_t - R_t = \dfrac{D_t}{P_t}\dfrac{P_t}{P_{t-1}} = \dfrac{D_t}{P_t}(1 + R_t)$.
- Dividing both sides by $1 + R_t$ and the proof of (8) is complete. □

Example 4.10

Example 6.3.

- Suppose we can observe the simple and total returns of a stock, but we do not have information about the dividends.
- Specifically, on day t, the simple return is 1% and the total return is 1.9%.
- What is the (implied) **dividend yield**?
- Applying (8), we obtain

$$\frac{1.9\% - 1\%}{1 + 1\%} = \frac{0.9\%}{101\%} = 0.0089 = 0.89\%.$$

Example 4.11

Example 6.4.

- For the four dividends in 2017 in Slide 44, the end of day stock prices of Coca Cola a day before ex dates are, respectively, \$42.03 (March 13), \$45.03 (June 13), \$46.11 (September 14), and \$45.77 (November 30).
- From Slide 44, each dividend cash amount is \$0.37.
- Therefore, the annual dividend yield is

$$0.37 \cdot \left(\frac{1}{42.03} + \frac{1}{45.03} + \frac{1}{46.11} + \frac{1}{45.77} \right) = 3.31\%.$$

Another Way to Compute Dividend Yield

- Suppose today's date is June 30, 2017.
- A **backward-looking dividend yield** is to take the four most recent dividend payments before June 30, namely, two dividends of \$0.35 each in the second half of 2016, and two dividends of \$0.37 each in the first half of 2017.
- Given that the stock price of Coca-Cola company is \$44.85 on June 30, 2017, the dividend yield is obtained as $2\times(\$0.35 + \$0.37)/\$44.85 = 3.21\%$.
- An implicit assumption in this approach of computing the dividend yield is that investors are holding the stock for at least a year.

Why Adjust?

- Why do we want to adjust stock prices?
- First and foremost, it is at times imperative to analyze total return, taking into account dividend re-investments for reporting performance and so on.
- Second and equally important, we may need to apply trading strategies based on the time series of stock prices.

Reinvestment

- Suppose we receive the dividend D_t and we immediately **reinvest** this D_t into the same stock.

- Suppose we initially have N shares. The total dividend amount we receive in dollars is ND_t.

- With this amount of cash, we can buy $\dfrac{ND_t}{P_t}$ shares. You have just transformed the cash dividend into shares.

- So at the end of time t, our number of shares has increased from N to $N\left(1+\dfrac{D_t}{P_t}\right)$.

Dividend and Total Return

- Suppose we can *hypothetically* liquidate your entire position.
- Let us calculate our return on paper:

$$\check{R}_t = \frac{N\left(1+\dfrac{D_t}{P_t}\right)P_t - NP_{t-1}}{NP_{t-1}} \tag{9}$$

$$= \frac{P_t + D_t - P_{t-1}}{P_{t-1}} \tag{10}$$

$$= \frac{P_t - P_{t-1}}{P_{t-1}} + \frac{D_t}{P_{t-1}}. \tag{11}$$

- The second equality (10) is exactly the same as our earlier definition of **total return**, which is (7).

Actual versus On Paper

- In reality, of course, we do not receive dividend cash on ex date t per se.
- What we can do, nevertheless, is to borrow money equivalent to ND_t, and use that amount of cash to reinvest, i.e., transform from cash into owning more shares on the same stock.
- Since we are entitled to receive ND_t on the date of payment, we are able to repay the bank. Obviously, we need to pay interest to the lending bank.
- In all the definitions, we are not taking all the transaction costs into account. We also ignore the interest paid.
- So the **actualized total return** is smaller than the **total return on paper**.

Backward Adjustment

Definition 7.1.

Knowing the ex date t and the dividend per share D_t, the **dividend adjustment factor** is defined as

$$B_t := \frac{1}{1 + \dfrac{D_t}{P_t}}.$$

The adjusted stock price for $s = t-1, t-2, \ldots$ 2, 1, 0 is defined as

$$P_{b,s} = P_s B_t.$$

Backward Adjustment

Proposition 4

The total return $\check{R}_t$ can be expressed in terms of the adjusted price as follows:

$$\check{R}_t = \frac{P_t - P_{b,t-1}}{P_{b,t-1}}. \quad (12)$$

- It has the same form of a simple return.
- Note that P_t is the ex date stock price and thus it needs no adjustment.

Proof of Proposition 4

Proof.

- We multiply the total return (9) by $1 = B_t/B_t$ to obtain

$$\check{R}_t = \frac{N\left(1+\dfrac{D_t}{P_t}\right)P_tB_t - NP_{t-1}B_t}{NP_{t-1}B_t} = \frac{NP_t - NP_{b,t-1}}{NP_{b,t-1}} = \frac{P_t - P_{b,t-1}}{P_{b,t-1}}.$$

- In other words, to compute total return, we need to use the time series of adjusted prices.

□

Return on Non Ex Date

- Moreover, if neither s nor $s-1$ is an ex date, it can be easily shown that

$$\breve{R}_s = \frac{P_{b,s} - P_{b,s-1}}{P_{b,s-1}} = \frac{P_s B_t - P_{s-1} B_t}{P_{s-1} B_t} = \frac{P_s - P_{s-1}}{P_{s-1}} = R_s.$$

- This result is consistent with (8). Since $D_s = 0$, it must be that $\breve{R}_s = R_s$. Also, it is important to emphasize that for any arbitrary non-zero number α,

$$R_t = \frac{\alpha P_t - \alpha P_{t-1}}{\alpha P_{t-1}} = \frac{P_t - P_{t-1}}{P_{t-1}}.$$

- In other words, using adjusted prices on days that do not involve any dividend payment at all will produce the same value for simple return as using the unadjusted prices. It also follows that the log return will not be affected by the adjustment factor when there is no dividend payment.

Backward Adjustment Algorithm

The algorithm for adjusting the stock prices backward works as follows:

1. Compute all the n dividend adjustment factors B_{t_i}, where $i = 1, 2, \ldots, n$.
2. For all the oldest prices before the first ex date t_1, multiply them by B_{t_1}.
3. For all the oldest prices before t_2, multiply them by B_{t_2}.
4. Do likewise for $i = 3, 4, \ldots, n$.
5. For the most recent prices from t_n onward, no adjustment is needed.

How the Backward Adjustment is

- The outcome is that the prices before t_1 are multiplied by all the dividend adjustment factors, i.e.,

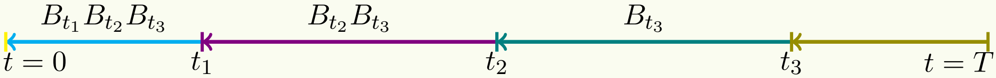
$$P_{b,s} = B_{t_1} \times B_{t_2} \times \cdots \times B_{t_n} \times P_s, \quad \text{for } s = 0, 1, 2, \ldots, t_1 - 1.$$

- For stock prices between t_1 and $t_2 - 1$, they are adjusted as follows:

$$P_{b,s} = B_{t_2} \times B_{t_3} \times \cdots \times B_{t_n} \times P_s, \quad \text{for } s = t_1, t_1 + 1, t_1 + 2, \ldots, t_2 - 1.$$

- In general, with $t_0 = 0$, and for $j = 1, 2, \ldots, n$, the adjusted prices are given by

$$P_{b,s} = B_{t_j} \times B_{t_{j+1}} \times \cdots \times B_{t_n} \times P_s, \quad \text{for } s = t_{j-1}, t_{j-1} + 1, t_{j-1} + 2, \ldots, t_j - 1.$$



Forward Adjustment

Definition 7.2.

The **forward dividend adjustment factor** F_t is defined as

$$F_t := \frac{1}{B_t} = 1 + \frac{D_t}{P_t}, \tag{13}$$

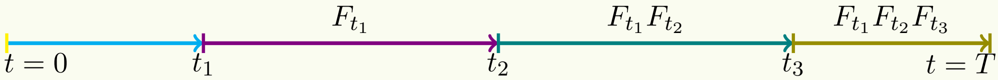
where t is the ex date. The **total-return stock prices** are given by

$$P_{f,s} := P_s F_t, \quad \text{for } s = t, t+1, t+2, \ldots.$$

- If we invest \$1,000 today, what is a reasonable estimate of the total value (before costs) of our investment in the future, at least on paper?
- Forward adjustment is suitable for long-term investors such as GIC.

Forward Dividend Adjustment Factor Algorithm

1. Calculate all the n forward dividend adjustment factors F_{t_i}, where $i = 1, 2, \ldots, n$,
2. Start from chronologically the oldest date, i.e., $t = 0$.
3. Do not adjust the stock prices before the first ex date t_1.
4. Multiply by F_{t_1} all prices from P_{t_1} through P_T.
5. Multiply by F_{t_2} all prices from P_{t_2} through P_T.
6. Do likewise for $i = 3, 4, \ldots, n$.



Outcome from Forward Adjustment

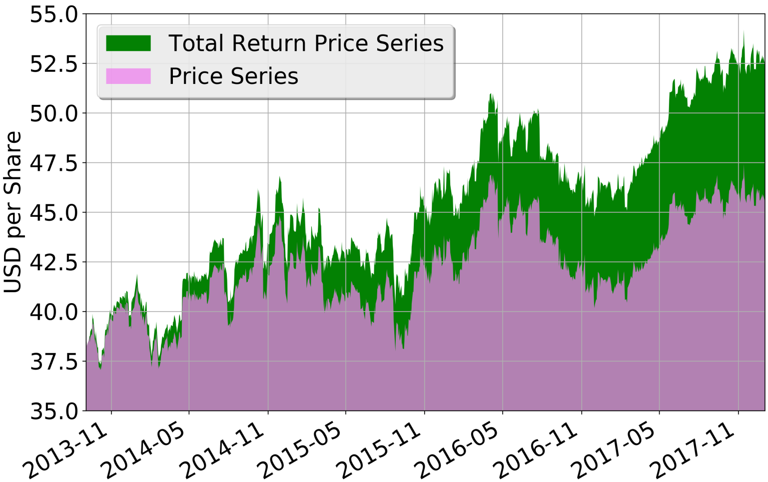
- The outcome from forward adjustment is that from the most current ex date t_n to the most current time T, the stock prices within and inclusive of these two dates get adjusted by all the forward dividend adjustment factors F_{t_i}.
- In other words,

$$P_{f,s} = \prod_{i=1}^{n} F_{t_i} P_s, \quad \text{for } s = t_n,\, t_n + 1, \ldots, T.$$

- Generally, when $k < n$,

$$P_{f,s} = \prod_{i=1}^{k} F_{t_i} P_s, \quad \text{for } s = t_k,\, t_{k-1} + 1, \ldots, t_{k+1} - 1.$$

Prices and Total-Return Prices of Coca-Cola



Insights

- Prices and index levels are non-stationary whereas returns are stationary.
- There are several definitions of return on an asset.
- Simple return is always larger than the log return.
- Geometric average return is always larger than the arithmetic average of the log return.
- What matters to the investor is the geometric return.
- In the absence of any company news, the stock price should drop by an amount equal to the dividend per share on ex date.
- You can back out the dividend yield with the total return and the simple return.
- The total return can be expressed in terms of the dividend-adjusted price.