# Week 3
# Determinant

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# Learning Outcomes

- Provide the reasons for why we need to study determinant.
- Elaborate important vocabulary and concepts such as determinant, cyclic, interchange, identity, inverse, even & odd permutations, sign of permutation, co-factor, minor, adjoint matrix, and Cramer's formula.
- Connect key concepts and results to the takeaways of the previous lessons, especially with regards to the system of linear equations, Gauss-Jordan elimination method, basic row operations, basic matrices and so on.
- Determine whether a square matrix is regular or not.
- Solve the problems that involve the system of linear equations, which requires the computation of determinant.

# What is a determinant?

**Definition 1.1 (Two-Dimensional Matrix's Determinant).**

With respect to $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we define its determinant as

$$\det \boldsymbol{A} = a_{11}a_{22} - a_{12}a_{21},$$

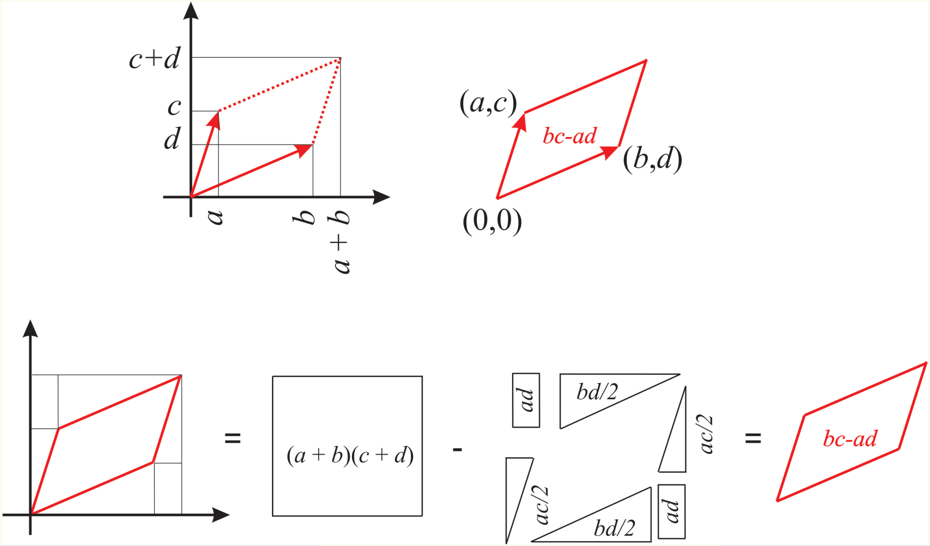
which is a scalar, and we write

$$\det \boldsymbol{A} =: \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Note that the inverse matrix is

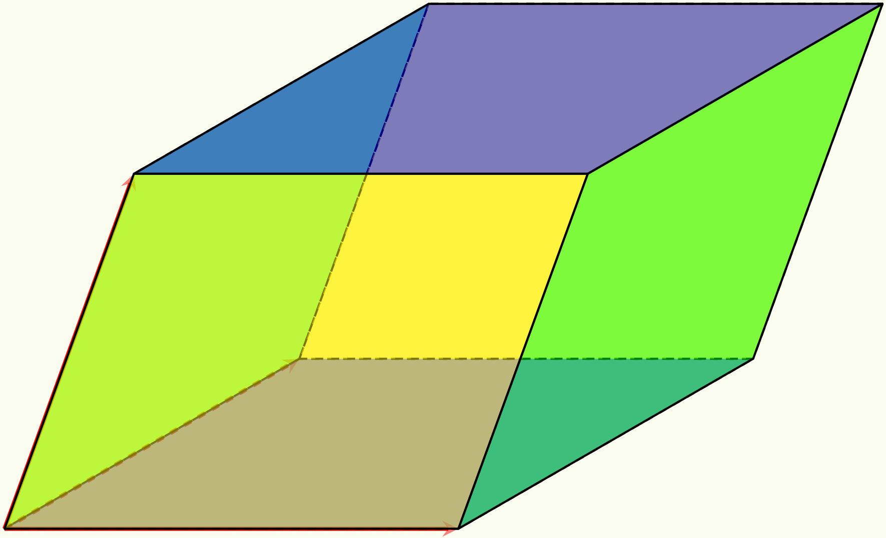
$$\boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

# Geometrical Interpretation




$$\begin{vmatrix} b & a \\ d & c \end{vmatrix}$$

# Illustration of 3-Dimensional Determinant



# Simple Solution

- Two-variable system of linear equations

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases} \iff \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},$$

where $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

- The solution is $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$, which can be written in terms of only the determinants.

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

# Humor

On this quiz, I asked you to find the determinant of a 2x3 matrix. Some of you, to my great amusement, actually tried to do this.

— **Math Professor Quote**

# Definition of Permutation

**Definition 2.1 (Permutation).**

Let $S := \{1, 2, \ldots, n\}$ be a set of $n$ natural numbers. A permutation is defined as a 1-to-1 mapping $\sigma$ from $S$ to $S$ such that

$$\sigma(1) = k_1,\ \sigma(2) = k_2,\ \ldots,\ \sigma(n) = k_n,$$

where $\{k_1, k_2, \ldots, k_n\}$ is a re-arranged version of set $S$. Permutation at times is expressed as

$$\sigma := \begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}. \tag{1}$$

**Definition 2.2 (Product of Two Permutations).**

Let $\sigma$ and $\tau$ be two permutations on $S$. The product $\sigma\tau(i)$ is defined as $\sigma\big(\tau(i)\big)$ for $i = 1, 2, \ldots, n$.

# Examples

(1) When $\sigma(1) = 3, \sigma(2) = 2, \sigma(3) = 4$, and $\sigma(4) = 1$, we write

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 3 & 1 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 1 \\ 1 & 4 & 3 \end{pmatrix}.$$

(2) Consider $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

- $\sigma\tau$ and $\tau\sigma$ are given by

$$\begin{aligned} \sigma(\tau(1)) &= \sigma(3) = 1 & \tau(\sigma(1)) &= \tau(2) = 2, \\ \sigma(\tau(2)) &= \sigma(2) = 3 & \tau(\sigma(2)) &= \tau(3) = 1, \\ \sigma(\tau(3)) &= \sigma(1) = 2 & \tau(\sigma(3)) &= \tau(1) = 3. \end{aligned}$$

- Thus, $\sigma\tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ and $\tau\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$.
- Takeaway: $\sigma\tau \neq \tau\sigma$, i.e. non-commutative.

# Identity, Inverse, and Cyclic Permutations

## Definition 2.3 (Identity and Inverse Permutation).

The identity permutation is defined as $\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$. The inverse permutation is defined as $\sigma^{-1} := \begin{pmatrix} k_1 & k_2 & \cdots & k_n \\ 1 & 2 & \cdots & n \end{pmatrix}$.

## Definition 2.4 (Cyclic Permutation).

From a set of $n$ numbers $\{1, 2, \ldots, n\}$, take $r \leqq n$ different numbers denoted by $k_1, k_2, \ldots, k_r$. The cyclic permutation is defined as

$$\sigma := \begin{pmatrix} k_1 & k_2 & \cdots & k_{r-1} & k_r \\ k_2 & k_3 & \cdots & k_r & k_1 \end{pmatrix}.$$

That is, the sequence of $r$ different numbers is shifted by one position to the right.

# Examples

(1) The inverse permutation of $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$ is

$$\sigma^{-1} = \begin{pmatrix} 4 & 2 & 1 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

(2) Suppose $\sigma(2) = 5, \sigma(5) = 3, \sigma(3) = 2$, and the other numbers do not permute. Then we can write the cyclic permutation simply as $(5 \quad 2 \quad 3)$. It can also be written as $(2 \quad 3 \quad 5)$, or as $(3 \quad 5 \quad 2)$.

(3) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 5 & 7 & 2 & 4 & 6 & 8 \end{pmatrix}$.

- Numbers 1 and 8 do not permute.
- Since $\sigma(2) = 3, \sigma(3) = 5, \sigma(5) = 2$, we have a cyclic permutation $(2 \ \ 3 \ \ 5)$.
- Since $\sigma(4) = 7, \sigma(7) = 6, \sigma(6) = 4$, we have another cyclic permutation $(4 \ \ 7 \ \ 6)$.
- Hence $\sigma = (2 \ \ 3 \ \ 5)(4 \ \ 7 \ \ 6)$.

# Interchanges and Sign

**Definition 2.5 (Interchange Permutation).**

An interchange is defined as the cyclic permutation for 2 elements in the set $S$. while keeping other elements unchanged. We denote the interchange of $i$ and $j$ as $(i\ j)$.

**Theorem 2.6 (Product of Cyclic Permutations or Interchanges).**

1. *Any permutation can be expressed as the product of cyclic permutations.*
2. *Any permutation can be expressed as the product of interchanges.*

**Definition 2.7 (Sign of Permutation).**

When the permutation $\sigma$ is expressed as a product of $m$ interchanges, the sign of the permutation is defined as

$$\mathrm{sgn}(\sigma) := (-1)^m.$$

The sign of the identity permutation is 1 (i.e., $m = 0$).

# Even and Odd Permutations

**Definition 2.8 (Even and Odd Permutations).**

When $\mathrm{sgn}(\sigma) = 1$, the permutation is said to be even. On the other hand, if $\mathrm{sgn}(\sigma) = -1$, the permutation is said to be odd.

**Theorem 2.9 (Uniqueness of Sign of a Permutation).**

*When the permutation $\sigma$ is expressed as $\sigma := \tau_1 \cdots \tau_k$, its sign does not depend on whether $k$ is even or odd. The sign of $\sigma$ is uniquely determined.*

**Definition 2.10 (Set of All Permutations).**

The set of all possible permutations involving $n$ numbers is denoted by $S_n$.

# Cyclic Permutation as a Product of Interchanges

- Note that for any interchange, $(i \;\; j) = (j \;\; i)$

- Example: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

★ As a product of 3 interchanges: $\sigma = (1 \;\; 2)(1 \;\; 3)(1 \;\; 4)$

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 1 & 1 \\ & \downarrow & \downarrow & \\ & 3 & 4 & \end{array}$$

★ $\operatorname{sgn}(\sigma) = (-1)^3 = -1.$

# Definition of Determinant

**Definition 3.1 (Definition of a Generic Determinant).**

Given a $n$-dimensional square matrix $\boldsymbol{A} = [a_{ij}]$, its determinant is defined as

$$\det \boldsymbol{A} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

- Other notations of determinant:

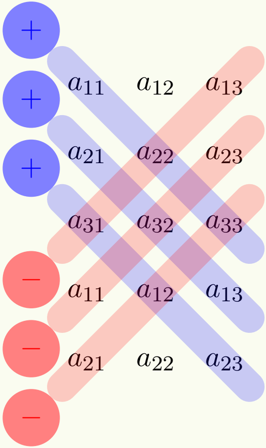
$$|\boldsymbol{A}|, \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}, \quad \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \det \begin{bmatrix} \boldsymbol{a}_1' \\ \boldsymbol{a}_2' \\ \vdots \\ \boldsymbol{a}_n' \end{bmatrix}, \quad \begin{vmatrix} \boldsymbol{a}_1' \\ \boldsymbol{a}_2' \\ \vdots \\ \boldsymbol{a}_n' \end{vmatrix},$$

$$\begin{vmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{vmatrix},$$

where $\boldsymbol{a}_i, i = 1, 2, \ldots, n$, is a $n$-dimensional vector.

# Illustration of Determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \left(a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}\right) - \left(a_{13}a_{22}a_{31} + a_{23}a_{32}a_{11} + a_{33}a_{12}a_{21}\right)$$



# Another Illustration

$$\begin{array}{l} \qquad\qquad\;\; + \qquad + \qquad +\,- \quad - \qquad - \\ \boldsymbol{A}_{3\times 3} = \left[\begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right]\begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{array} \end{array}$$

$$\det \boldsymbol{A} = (a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3) - (a_3 b_2 c_1 + b_3 c_2 a_1 + c_3 a_2 b_1)$$

# Theorem 3.2

## Theorem 3.2.

*The following equations are true.*

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & \cdots & a_{1\,n-1} & 0 \\ \vdots & \ddots & \vdots & 0 \\ a_{n-1\,1} & \cdots & a_{n-1\,n-1} & \vdots \\ a_{n1} & \cdots & a_{n\,n-1} & a_{nn} \end{vmatrix} = a_{nn} \begin{vmatrix} a_{11} & \cdots & a_{1\,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1\,1} & \cdots & a_{n-1\,n-1} \end{vmatrix}$$

# Theorems 3.3 and 3.4

## Theorem 3.3 (Invariance under Transpose).

*For a square matrix $\boldsymbol{A}$,*

$$\det \boldsymbol{A}' = \det \boldsymbol{A}.$$

## Theorem 3.4 (Scalar Multiple and Addition).

*Let $\boldsymbol{a}_i$ and $\boldsymbol{b}_i, i = 1, 2, \ldots n$ be 1-dimensional vectors and $c$ a scalar. Then*

$$\begin{vmatrix} \boldsymbol{a}_1' \\ \vdots \\ c\boldsymbol{a}_i' \\ \vdots \\ \boldsymbol{a}_n' \end{vmatrix} = c \begin{vmatrix} \boldsymbol{a}_1' \\ \vdots \\ \boldsymbol{a}_i' \\ \vdots \\ \boldsymbol{a}_n' \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \boldsymbol{a}_1' \\ \vdots \\ \boldsymbol{a}_i' + \boldsymbol{b}_i' \\ \vdots \\ \boldsymbol{a}_n' \end{vmatrix} = \begin{vmatrix} \boldsymbol{a}_1' \\ \vdots \\ \boldsymbol{a}_i' \\ \vdots \\ \boldsymbol{a}_n' \end{vmatrix} + \begin{vmatrix} \boldsymbol{a}_1' \\ \vdots \\ \boldsymbol{b}_i' \\ \vdots \\ \boldsymbol{a}_n' \end{vmatrix}.$$

# Theorem 3.5

**Theorem 3.5 (Interchangeability).**

$$(1)\quad \begin{vmatrix} \vdots \\ \boldsymbol{a}_i' \\ \vdots \\ \boldsymbol{a}_j' \\ \vdots \end{vmatrix} = -\begin{vmatrix} \vdots \\ \boldsymbol{a}_j' \\ \vdots \\ \boldsymbol{a}_i' \\ \vdots \end{vmatrix}, \qquad (2)\quad \begin{vmatrix} \vdots \\ \boldsymbol{a}_i' \\ \vdots \\ \boldsymbol{a}_i' \\ \vdots \end{vmatrix} = 0.$$

- Swapping one row with the other results in only a sign change.

# Theorem 3.6

## Theorem 3.6 (Invariance under a Basic Row Operation).

$$\begin{vmatrix} \vdots \\ \boldsymbol{a}_i' + c\boldsymbol{a}_j' \\ \vdots \\ \boldsymbol{a}_j' \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ \boldsymbol{a}_i' \\ \vdots \\ \boldsymbol{a}_j' \\ \vdots \end{vmatrix}$$

# Example: Application of Theorems

$$\begin{vmatrix} -4 & 1 & 1 \\ 1 & 2 & 0 \\ -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -4 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{vmatrix} \qquad (R_3 \leftarrow R_3 + R_2 \text{ [Th 3.6]})$$

$$= \begin{vmatrix} 0 & 9 & 1 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{vmatrix} \qquad (R_1 \leftarrow R_1 + 4R_2 \text{ [Th 3.6]})$$

$$= \begin{vmatrix} 0 & 0 & -2 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{vmatrix} \qquad (R_1 \leftarrow R_1 - 3R_3 \text{ [Th 3.6]})$$

$$= -\begin{vmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 3 & 1 \end{vmatrix} \qquad (R_1 \leftrightarrow R_2 \text{ [Th 3.5]})$$

$$= -\left(-\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{vmatrix}\right) \qquad (R_2 \leftrightarrow R_3 \text{ [Th 3.5]})$$

$$= \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{vmatrix} = 1 \times 3 \times (-2) = -6. \qquad (\text{[Th 3.2]})$$

# Theorem 3.7

**Theorem 3.7 (Determinants of Basic Operation Matrices).**

*For the basic operation matrices, scalars $c$ and $d$,*

$$\begin{aligned} \left|\boldsymbol{P}_{ij}\right| &= -\left|\boldsymbol{I}_n\right| = -1, & \left|\boldsymbol{P}_{ij}\boldsymbol{A}\right| &= \left|\boldsymbol{P}_{ij}\right|\,\left|\boldsymbol{A}\right|, \\ \left|\boldsymbol{Q}_i(c)\right| &= c\left|\boldsymbol{I}_n\right| = c, & \left|\boldsymbol{Q}_i(c)\boldsymbol{A}\right| &= \left|\boldsymbol{Q}_i(c)\right|\,\left|\boldsymbol{A}\right|, \\ \left|\boldsymbol{R}_{ij}(d)\right| &= \left|\boldsymbol{I}_n\right| = 1, & \left|\boldsymbol{R}_{ij}(d)\boldsymbol{A}\right| &= \left|\boldsymbol{R}_{ij}(d)\right|\,\left|\boldsymbol{A}\right|. \end{aligned}$$

# Theorem 3.9

**Theorem 3.8 (Regularity and Non-Zero Determinant).**

*An $n$-dimensional square matrix $\boldsymbol{A}$ is regular if and only if $\det \boldsymbol{A} \neq 0$.*

**Theorem 3.9 (Interchangeability with Multiplication ).**

*For the $n$-dimensional matrices $\boldsymbol{A}$ and $\boldsymbol{B}$,*

$$\left|\boldsymbol{A}\boldsymbol{B}\right| = \left|\boldsymbol{A}\right| \left|\boldsymbol{B}\right|.$$

# Application

- Suppose for some square matrix $\boldsymbol{P}$ such that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix with strictly positive diagonal elements $\lambda_1, \lambda_2, \cdots, \lambda_n > 0$.
- Applying Theorem 3.9, we find that

$$\left|\boldsymbol{A}\right| = \left|\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}\right| = \left|\boldsymbol{P}\right| \left|\boldsymbol{D}\right| \left|\boldsymbol{P}^{-1}\right| = \left|\boldsymbol{D}\right| \left|\boldsymbol{P}\boldsymbol{P}^{-1}\right| = \prod_{i=1}^{n} \lambda_i$$

- It follows that

$$\begin{aligned}\ln\left|\boldsymbol{A}\right| &= \sum_{i=1}^{n} \ln(\lambda_i) = \operatorname{tr}\ln(\boldsymbol{D}) = \operatorname{tr}\ln\left(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\right) \\ &= \operatorname{tr}\left(\ln\boldsymbol{A} + \ln\boldsymbol{P}^{-1} + \ln\boldsymbol{P}\right) = \operatorname{tr}\left(\ln\boldsymbol{A} + \ln\left(\boldsymbol{P}^{-1}\boldsymbol{P}\right)\right) \\ &= \operatorname{tr}\ln\boldsymbol{A}.\end{aligned}$$

- Therefore $\boxed{\ln\det = \operatorname{tr}\ln}$.

# Partitioned with a Null Sub-Matrix

**Theorem 4.1 (With Null and Identity Sub-Matrices).**

*Let $\boldsymbol{A}$ be an $n$-dimensional matrix.*

$(1)$ *If* $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{A}_{12} \\ \boldsymbol{0} & \boldsymbol{A}_{22} \end{bmatrix}$, *then* $\det \boldsymbol{A} = \det \boldsymbol{A}_{22}$.

$(2)$ *If* $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{0} \\ \boldsymbol{A}_{21} & \boldsymbol{I} \end{bmatrix}$, *then* $\det \boldsymbol{A} = \det \boldsymbol{A}_{11}$.

**Theorem 4.2 (With Null Sub-Matrice Only).**

*If an $n$-dimensional square matrix can be partitioned as* $\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{0} & \boldsymbol{A}_{22}. \end{bmatrix}$, *or* $\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{0} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22}. \end{bmatrix}$, *then*

$$\det \boldsymbol{A} = \big(\det \boldsymbol{A}_{11}\big)\big(\det \boldsymbol{A}_{22}\big).$$

# The Schur Complement

## Definition 4.3 (Schur's Complement).

Let $M$ be a partitioned matrix of the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A$ and $D$ are square matrices.

- If $A$ is invertible, Shur's complement of $A$ in $M$ is defined as

$$M_A := D - CA^{-1}B.$$

- If $D$ is invertible, Shur's complement of $D$ in $M$ is defined as

$$M_D := A - BD^{-1}C.$$

# Schur's Decomposition of a Partitioned Matrix

**Theorem 4.4 (Schur's Decomposition).**

*Let $M$ be a partitioned matrix of the form*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

*where $A$ and $D$ are square matrices.*

- *If $A$ is invertible, then*
$$M = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & M_A \end{bmatrix} \begin{bmatrix} I & AB^{-1} \\ 0 & I \end{bmatrix}. \tag{2}$$

- *If $D$ is invertible, then*
$$M = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} M_D & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}. \tag{3}$$

# Proof of Schur's Decomposition

- The product of the three matrices on the right-hand side of (2) can be rewritten as

$$
\begin{aligned}
M &= \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & M_A \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} AI + 00 & AA^{-1}B + 0I \\ 0I + (D - CA^{-1}B)0 & 0A^{-1}B + (D - CA^{-1}B)I \end{bmatrix} \\
&= \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix} = \begin{bmatrix} IA + 00 & IB + 0(D - CA^{-1}B) \\ CA^{-1}A & CA^{-1}B + I(D - CA^{-1}B) \end{bmatrix} \\
&= \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\end{aligned}
$$

- The proof of (3) proceeds by the same matrix multiplication.

# Determinant of Partitioned Matrix

**Theorem 4.5 (Determinant of Partitioned Matrix).**

*Let $M$ be a partitioned matrix of the form*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

*where $A$ and $D$ are square matrices.*

- *If $A$ is invertible, then*

$$\det(\boldsymbol{M}) = \det(\boldsymbol{A}) \det\left(\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B}\right) \tag{4}$$

- *If $D$ is invertible, then*

$$\det(\boldsymbol{M}) = \det(\boldsymbol{D}) \det\left(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}\right) \tag{5}$$

# Proof of the Determinant of Partitioned Matrix

- We apply the Schur decomposition to $\boldsymbol{M}$. It is clear that

$$\begin{vmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{C}\boldsymbol{A}^{-1} & \boldsymbol{I} \end{vmatrix} = 1 \qquad \text{and} \qquad \begin{vmatrix} \boldsymbol{I} & \boldsymbol{A}^{-1}\boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{I} \end{vmatrix} = 1$$

- For the middle matrix in the Schur decomposition,

$$\begin{vmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B} \end{vmatrix} = \det(\boldsymbol{A}) \det\left(\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B}\right).$$

- The proof of (5) proceeds by the same matrix multiplication.

# Weinstein-Aronszajn Identity

## Theorem 4.6 (Weinstein-Aronszajn Identity).

*Let $A$ be a $p$ by $q$ matrix and $B$ a $q$ by $p$ matrix. Then*

$$\det\left(\boldsymbol{I}_p + \boldsymbol{A}\boldsymbol{B}\right) = \det\left(\boldsymbol{I}_q + \boldsymbol{B}\boldsymbol{A}\right),$$

## Proof.

- Let $\boldsymbol{M}$ be a matrix comprising the four partitions $\boldsymbol{I}_p$, $\boldsymbol{B}$, and $\boldsymbol{I}_q$.
$$\boldsymbol{M} := \begin{bmatrix} \boldsymbol{I}_p & -\boldsymbol{A} \\ \boldsymbol{B} & \boldsymbol{I}_q \end{bmatrix}$$
- Since $\boldsymbol{I}_p$ is invertible, the formula for the determinant of a partitioned matrix gives $\det(\boldsymbol{M}) = \det(\boldsymbol{I}_p)\det\left(\boldsymbol{I}_q - \boldsymbol{B}\boldsymbol{I}_p^{-1}(-\boldsymbol{A})\right) = \det\left(\boldsymbol{I}_q + \boldsymbol{B}\boldsymbol{A}\right)$.
- Since $\boldsymbol{I}_q$ is invertible, the formula for the determinant of a partitioned matrix gives $\det(\boldsymbol{M}) = \det(\boldsymbol{I}_q)\det\left(\boldsymbol{I}_p - (-\boldsymbol{A})\boldsymbol{I}_q^{-1}\boldsymbol{B}\right) = \det\left(\boldsymbol{I}_p + \boldsymbol{A}\boldsymbol{B}\right)$.

□

# Applications of Weinstein-Aronszajn Identity

- The identity is useful to relate determinants of large matrices to determinants of smaller ones

- Suppose $\lambda$ is a non-zero number. It can be shown that

$$\det\left(\boldsymbol{AB} - \lambda \boldsymbol{I}_p\right) = (-\lambda)^{p-q} \det\left(\boldsymbol{BA} - \lambda \boldsymbol{I}_q\right).$$

- This identity is useful in developing a Bayes estimator for multivariate Gaussian distributions.

# Minor of a Matrix

- The **minors** of $\boldsymbol{A}$, each of size $(n-1)\times(n-1)$, is a resulting matrix $\widetilde{\boldsymbol{A}}_{ij}$, after column $j$ and row $i$ have been deleted.

$$
\begin{array}{ccc}
 & \begin{array}{c}\text{column}\\ j \\ \downarrow\end{array} & \\
\widetilde{\boldsymbol{A}}_{ij} := & \begin{bmatrix}
a_{11} & \cdots & a_{1\,j-1} & a_{1\,j+1} & \cdots & a_{1\,n} \\
\vdots & & \vdots & \vdots & & \vdots \\
a_{i-1\,1} & \cdots & a_{i-1\,j-1} & a_{i-1\,j+1} & \cdots & a_{i-1\,n} \\
 & & & & & \\
a_{i+1\,1} & \cdots & a_{i+1\,j-1} & a_{i+1\,j+1} & \cdots & a_{i+1\,n} \\
\vdots & & \vdots & \vdots & & \vdots \\
a_{n\,1} & \cdots & a_{n\,j-1} & a_{n\,j+1} & \cdots & a_{n\,n}
\end{bmatrix} & \leftarrow \text{Row } i
\end{array}
$$

# Co-Factor and Expansion

**Definition 5.1 (Co-Factor).**

Given an $n$-dimensional square matrix $\boldsymbol{A} = \left[a_{ij}\right]$, its $(ij)$ co-factor $\widetilde{a}_{ij}$ is a scalar defined by the following formula:

$$\widetilde{a}_{ij} := (-1)^{i+j} \det \widetilde{\boldsymbol{A}}_{ij}.$$

**Theorem 5.2 (Expansion with Co-Factors).**

*For an $n$-dimensional matrix $\boldsymbol{A}$, its determinant can be written as an expansion of its co-factors with respect to the $i$-th row*

$$\det \boldsymbol{A} = a_{i1}\widetilde{a}_{i1} + \cdots + a_{i\,n}\widetilde{a}_{i\,n}, \qquad \textit{where } i = 1, 2, \ldots, n,$$

*or with respect to the $j$-th column*

$$\det \boldsymbol{A} = a_{1j}\widetilde{a}_{1j} + \cdots + a_{1\,j}\widetilde{a}_{n\,j}, \qquad \textit{where } j = 1, 2, \ldots, n. \tag{6}$$

# Example 5.3

## Example 5.3 (Expansion).

Compute the determinant of $\begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 2 & -1 & -4 \end{bmatrix}$ by expanding along the second column.

Answer:

$$\begin{aligned}
\begin{vmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 2 & -1 & -4 \end{vmatrix} &= (-1)^{1+2}(0)\begin{vmatrix} 3 & 5 \\ 2 & -4 \end{vmatrix} + (-1)^{2+2}(4)\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix} \\
&\quad +(-1)^{3+2}(-1)\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} \\
&= -32 + (-1) = -33.
\end{aligned}$$

# Co-Factor Matrix

**Definition 5.4 (Co-Factor or Adjoint Matrix).**

Given an $n$-dimensional square matrix $\boldsymbol{A} = [a_{ij}]$, we define $\widetilde{\boldsymbol{A}} := [\widetilde{a}_{ij}]$ and call its transpose $\widetilde{\boldsymbol{A}}'$ as the co-factor matrix or the adjoint matrix.

**Theorem 5.5 (Product with Co-Factor Matrix).**

*For an $n$-dimensional square matrix $\boldsymbol{A}$ and its co-factor $\widetilde{\boldsymbol{A}}'$, the following equations hold*

$$\boldsymbol{A}\widetilde{\boldsymbol{A}}' = \widetilde{\boldsymbol{A}}'\boldsymbol{A} = (\det \boldsymbol{A})\,\boldsymbol{I}.$$

**Theorem 5.6 (Inverse of a Matrix).**

*If an $n$-dimensional square matrix $\boldsymbol{A}$ is regular, then its inverse is given by*

$$\boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}}\widetilde{\boldsymbol{A}}'.$$

# Example 5.7

## Example 5.7 (Adjoint Matrix and Inverse).

Find the adjoint matrix of $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ and $\boldsymbol{A}^{-1}$.

- The co-factors are

$$\widetilde{a}_{11} = (-1)^{1+1} \begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix} = -7,\ \widetilde{a}_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1,\ \widetilde{a}_{13} = (-1)^{1+3} \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = 4,$$

$$\widetilde{a}_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -1,\ \widetilde{a}_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1,\ \widetilde{a}_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1,$$

$$\widetilde{a}_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} = 11,\ \widetilde{a}_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 2,\ \widetilde{a}_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} = -5.$$

# Example 5.7 (つづき)

- Consequently, the adjoint matrix $\boldsymbol{A}^\dagger \equiv \widetilde{\boldsymbol{A}}'$ is

$$\boldsymbol{A}^\dagger = \begin{bmatrix} -7 & -1 & 4 \\ -1 & -1 & 1 \\ 11 & 2 & -5 \end{bmatrix}' = \begin{bmatrix} -7 & -1 & 11 \\ -1 & -1 & 2 \\ 4 & 1 & -5 \end{bmatrix}.$$

- The determinant of $\boldsymbol{A}$ is

$$\begin{aligned} \det(\boldsymbol{A}) &= 1 \cdot (-3) \cdot 2 + 1 \cdot 1 \cdot 3 + 2 \cdot 1 \cdot 1 - \big(3 \cdot (-3) \cdot 1 + 1 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 2\big) \\ &= -1 - (-4) = 3. \end{aligned}$$

- Therefore, from Theorem 5.6,

$$\boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}} \boldsymbol{A}^\dagger = \frac{1}{3} \begin{bmatrix} -7 & -1 & 11 \\ -1 & -1 & 2 \\ 4 & 1 & -5 \end{bmatrix}.$$

# Cramer's Formula

**Theorem 5.8 (Cramer's Formula).**

*To solve the simultaneous first-order system of equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ for which the coefficient matrix $\boldsymbol{A}$ is regular, we have*

$$x_j = \frac{\det \boldsymbol{B}_j}{\det \boldsymbol{A}}, \tag{7}$$

*where the matrix $\boldsymbol{B}_j$ is given by*

$$\boldsymbol{B}_j := \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_{j-1} & \boldsymbol{b} & \boldsymbol{a}_{j+1} & \cdots & \boldsymbol{a}_n \end{bmatrix}.$$

*That is, column $j$ of $\boldsymbol{A}$ is replaced by $\boldsymbol{b}$.*

# Proof of Cramer's Formula

- Since $\boldsymbol{A}$ is regular, $\boldsymbol{A}^{-1}$ exists. Multiply $\boldsymbol{A}^{-1}$ from the left to both sides of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ to obtain the solution: $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$.
- Applying Theorem 5.6, we obtain $\boldsymbol{x} = \dfrac{1}{\det \boldsymbol{A}}\widetilde{\boldsymbol{A}}'\boldsymbol{b}$.
- The $j$-th element of the solution $\boldsymbol{x}$ is calculated as

$$x_j = \frac{1}{\det \boldsymbol{A}}\begin{bmatrix}\widetilde{a}_{1j} & \widetilde{a}_{2j} & \cdots & \widetilde{a}_{nj}\end{bmatrix}\begin{bmatrix}b_1 \\ b_2 \\ \vdots \\ b_n\end{bmatrix} = \frac{1}{\det \boldsymbol{A}}\left(b_1\widetilde{a}_{1j} + b_2\widetilde{a}_{2j} + \cdots + b_n\widetilde{a}_{nj}\right).$$

- With reference to (6), we obtain a co-factor expansion $b_1\widetilde{a}_{1j} + b_2\widetilde{a}_{2j} + \cdots + b_n\widetilde{a}_{nj}$, of the determinant of $\boldsymbol{B}_j$. Hence, we arrive at (7) and the proof is complete. $\square$

# Computing Determinants by Row Reduction

- Let $\boldsymbol{A}$ be a square matrix. After some number of row operations on $\boldsymbol{A}$, the row echelon matrix $\boldsymbol{B}$ is obtained. Then

$$\det \boldsymbol{A} = (-1)^r \frac{\text{product of the diagonal entries of } \boldsymbol{B}}{\text{product of scaling factors used}}, \tag{8}$$

where $r$ is the number of row swaps that have been performed.

- The recipe (8) is an efficient way of computing the determinant of a large matrix. The computational complexity of row reduction is $O(n^3)$, where $n$ is the dimension of the square matrix.

# Example

$$\begin{bmatrix} 0 & -7 & -4 \\ 2 & 4 & 6 \\ 3 & 7 & -1 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{bmatrix} 2 & 4 & 6 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{bmatrix} \xrightarrow{R_1 \longleftarrow \frac{1}{2} R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{bmatrix} \xrightarrow{R_3 \longleftarrow R_3 - 3R_1}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 0 & 1 & -10 \end{bmatrix} \xrightarrow{R_2 \longleftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & -7 & -4 \end{bmatrix} \xrightarrow{R_3 \longleftrightarrow R_3 + 7R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & 0 & -74 \end{bmatrix}$$

- We have made two row swaps and scaled once by a factor of $\frac{1}{2}$.
- So, the recipe (8) allows us to compute the determinant:

$$\det \begin{bmatrix} 0 & -7 & -4 \\ 2 & 4 & 6 \\ 3 & 7 & -1 \end{bmatrix} = (-1)^2 \frac{1 \cdot 1 \cdot (-74)}{\frac{1}{2}} = -148.$$

# Takeaways

- Determinant is a function that maps the space of real-valued square matrices into the set of real numbers.
- Doing a basic row operation on $\boldsymbol{A}$ does not change $\det(\boldsymbol{A})$.
- Scaling a row of $\boldsymbol{A}$ by a scalar $c$ multiplies the determinant by $c$.
- Swapping two rows of a matrix multiplies the determinant by $-1$.
- The determinant of the identity matrix $\boldsymbol{I}_n$ is equal to 1.
- Determinant is invariant under transpose and a basic matrix row operation.
- Co-factor approach provides a way to prove Cramer's formula, which is a system of linear equations by the ratios of determinants.
- The recipe approach to computing the determinant is more efficient than the co-factor approach.

# Keywords