Section 7
Double Integrals

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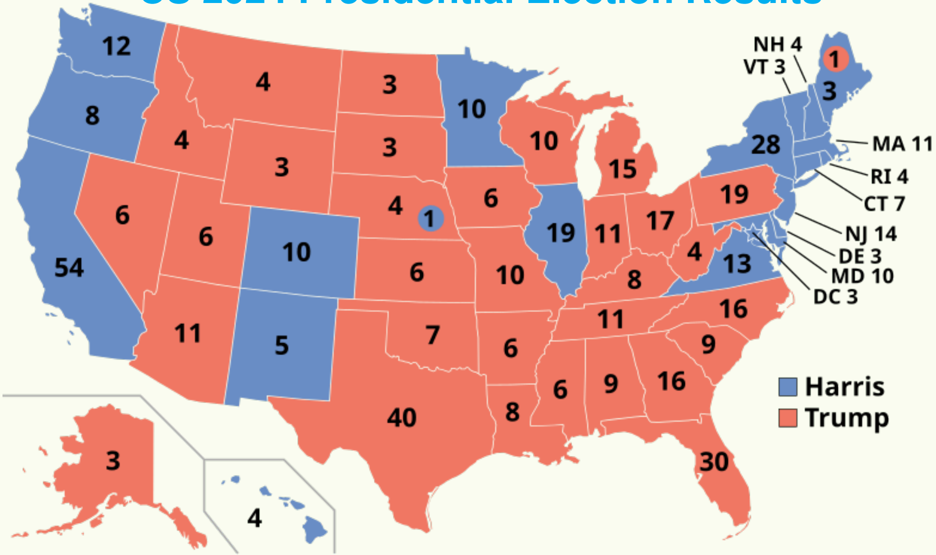
Table of Contents

Learning Outcomes

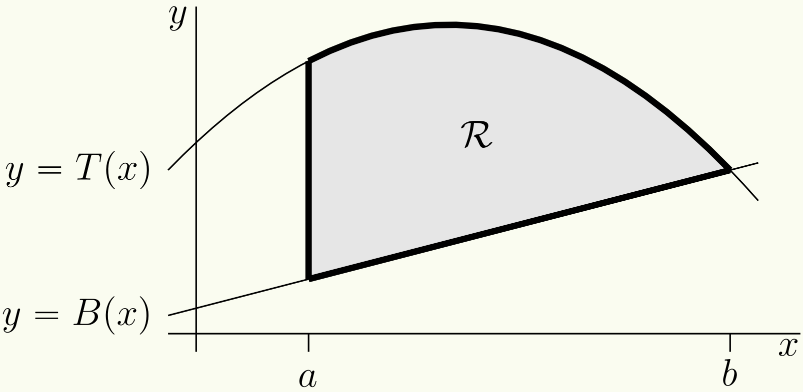
- Recall the concept of Riemann sum.
- Define the concepts of area density and mass.
- Construct through Riemann sum a double integral based on vertical slices.
- Construct through Riemann sum a double integral based on horizontal slices.
- Distinguish between inside and outside integrals, and elaborate the notion of "integrated out."
- Develop an intuitive understanding of Fubini's theorem.
- Analyze and apply the special case of double integral.

US 2024 Presidential Election Results



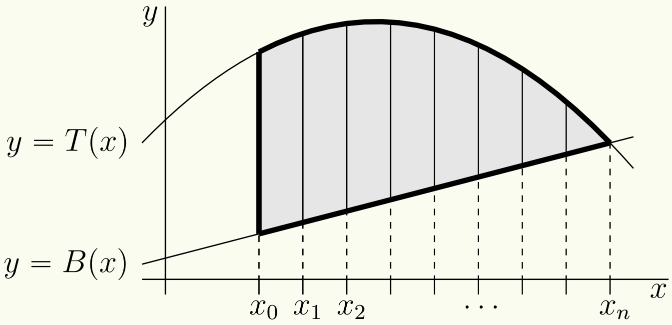
Density, Area, and Mass

$$R = \{ \ (x, y) \ | \ a \le x \le b, \ B(x) \le y \le T(x) \ \}$$



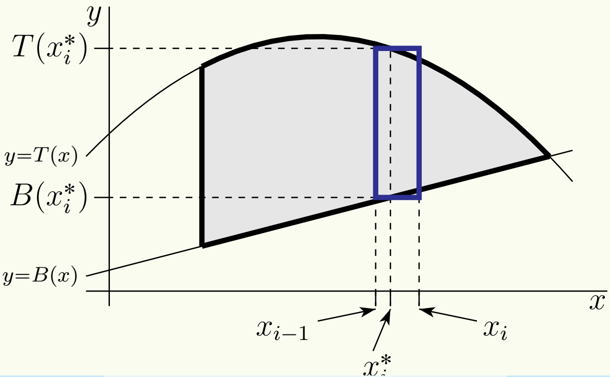
Vertical Slices

- Subdivide $\boldsymbol{R}$ into n narrow **vertical slices**, each of width $\Delta x = \dfrac{b-a}{n}$.
- Denote by $x_i = a + i\,\Delta x$ the x-coordinate of the right-hand edge of slice number i



Vertical Rectangle

- For each $i = 1, 2, \ldots, n$, slice number i has x running from x_{i-1} to x_i. We approximate its **area** by the area of a rectangle.
- We pick a number x_i^* between x_{i-1} and x_i and approximate the slice by a rectangle whose **top** is at $y = T(x_i^*)$ and whose **bottom** is at $y = B(x_i^*)$.



Definite Integral

- Thus the **area** of slice i is approximately $\big[T(x_i^*) - B(x_i^*)\big]\Delta x$.
- So the **Riemann sum** approximation of the area of $\boldsymbol{R}$ is

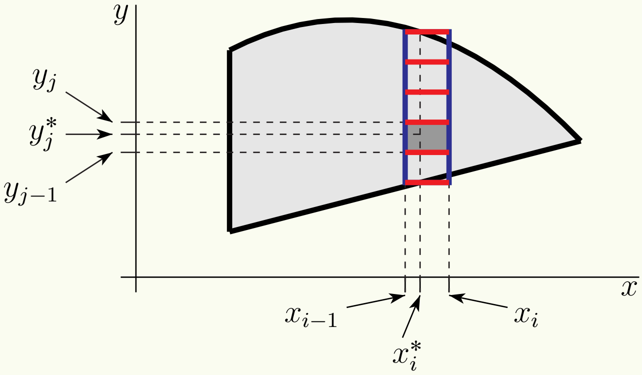
$$\text{Area} \approx \sum_{i=1}^{n} \big[T(x_i^*) - B(x_i^*)\big]\Delta x.$$

- By taking the **limit** as $n \to \infty$ (i.e. taking the limit as the width of the rectangles goes to zero), we convert the Riemann sum into a **definite integral** and at the same time our approximation of the area becomes the exact area:

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \big[T(x_i^*) - B(x_i^*)\big]\Delta x = \int_a^b \big[T(x) - B(x)\big]\,\mathrm{d}x.$$

Subdivision

- Subdivide slice number i into m tiny rectangles, each of width Δx and of height $\Delta y = \dfrac{1}{m}\left[T(x_i^*) - B(x_i^*)\right]$.
- Denote by $y_j = B(x_i^*) + j\,\Delta y$ the y-coordinate of the top of rectangle number j.



Approximation of Mass

- At this point we approximate the **density** inside each rectangle by a constant.
- For each $j = 1, 2, \ldots, m$, rectangle number j has y running from y_{j-1} to y_j. We pick a number y_j^* between y_{j-1} and y_j and approximate the density on rectangle number j in slice number i by the constant $f\left(x_i^*, y_j^*\right)$.
- Thus the mass of rectangle number j in slice number i is approximately $f\left(x_i^*, y_j^*\right) \Delta x \, \Delta y$.
- So the Riemann sum approximation of the mass of slice number i is

$$\textbf{Mass} \text{ of slice } i \approx \sum_{j=1}^{m} f\left(x_i^*, y_j^*\right) \Delta x \, \Delta y.$$

Note that the y_j^*'s depend on i and m.

Exact Mass

- By taking the limit as $m \to \infty$ (i.e. taking the limit as the height of the rectangles goes to zero), we convert the Riemann sum into a definite integral:

$$\text{Mass of slice } i \approx \Delta x \int_{B(x_i^*)}^{T(x_i^*)} f\left(x_i^*, y\right) \mathrm{d}y = F(x_i^*)\,\Delta x,$$

where

$$F(x) = \int_{B(x)}^{T(x)} f\left(x, y\right) \mathrm{d}y.$$

First Double Integral

- Notice that, while we started with the density $f(x,y)$ being a function of both x and y, by taking the limit of this Riemann sum, we have "**integrated out**" the dependence on y. As a result, $F(x)$ is a function of x only.
- Finally, taking the limit as $n \to \infty$ we get

$$\text{Mass} = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \int_{B(x_i^*)}^{T(x_i^*)} f\left(x_i^*, y\right)\, \mathrm{d}y = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*)\, \Delta x.$$

- Now we are back to our familiar 1-variable territory. The sum $\sum_{i=1}^{n} F(x_i^*)\, \Delta x$ is a Riemann sum approximation to the integral $\int_a^b F(x)\, \mathrm{d}x$. So

$$\text{Mass} = \int_a^b F(x)\, \mathrm{d}x = \int_a^b \left[\int_{B(x)}^{T(x)} f\left(x, y\right)\, \mathrm{d}y\right] \mathrm{d}x.$$

Notation for Top and Bottom

Iterated integral

$$\iint_{\boldsymbol{R}} f(x, y)\ \mathrm{d}x\ \mathrm{d}y = \int_a^b \left[\int_{B(x)}^{T(x)} f(x, y)\ \mathrm{d}y\right] \mathrm{d}x$$

$$= \int_a^b \int_{B(x)}^{T(x)} f(x, y)\ \mathrm{d}y\ \mathrm{d}x$$

$$= \int_a^b \mathrm{d}x \int_{B(x)}^{T(x)} \mathrm{d}y\, f(x, y).$$

Evaluation of $\int_a^b \int_{B(x)}^{T(x)} f(x,y)\ \mathrm{d}y\ \mathrm{d}x$

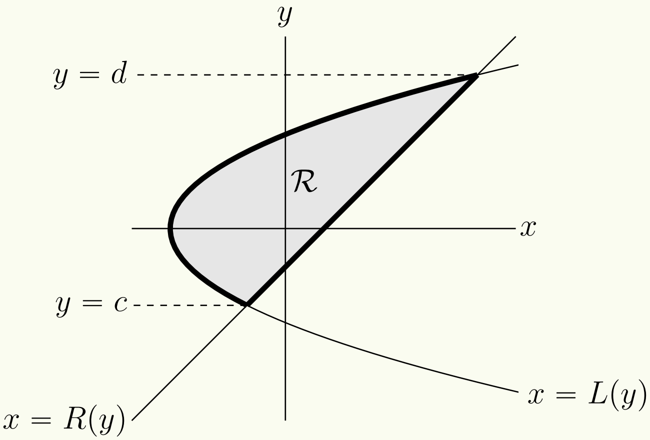
- First evaluate the inside integral $\int_{B(x)}^{T(x)} f(x,y)\ \mathrm{d}y$ using the inside limits of integration, and by treating x as a constant and using standard **single-variable integration** techniques.
- The result of the **inside integral** is a function of x only. Call it $F(x)$.
- Then evaluate the **outside integral** $\int_a^b F(x)\ \mathrm{d}x$, whose **integrand** is the answer to the inside integral.
- Again, this integral is evaluated using standard single-variable integration techniques.

Evaluation of $\int_a^b \mathrm{d}x \int_{B(x)}^{T(x)} \mathrm{d}y\, f(x,y)$

- First evaluate the inside integral $\int_{B(x)}^{T(x)} \mathrm{d}y\, f(x,y)$ using the limits of integration that are directly beside the $\mathrm{d}y$.
- Indeed the $\mathrm{d}y$ is written directly beside $\int_{B(x)}^{T(x)}$ to make it clear that the limits of integration $B(x)$ and $T(x)$ are for the y-integral.
- In the past you probably wrote this integral as $\int_{B(x)}^{T(x)} f(x,y)\ \mathrm{d}y$. The result of the inside integral is again a function of x only. Call it $F(x)$.
- Then evaluate the outside integral $\int_a^b \mathrm{d}x\, F(x)$, whose **integrand** is the answer to the inside integral and whose limits of integration are directly beside the $\mathrm{d}x$.

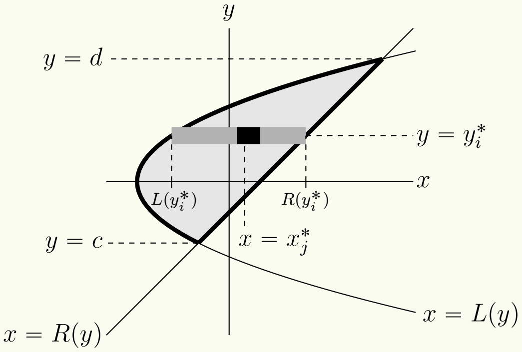
Region R

$$R = \left\{ \, (x, y) \,\middle|\, c \le y \le d, \; L(y) \le x \le R(y) \, \right\}$$



Horizontal Slice

- Subdivide the interval $c \le y \le d$ into n narrow subintervals, each of width $\Delta y = \dfrac{d-c}{n}$.
- We approximate slice number i by a thin horizontal rectangle. On this slice, the y-coordinate runs over a very narrow range. We pick a number y_i^*, somewhere in that range.

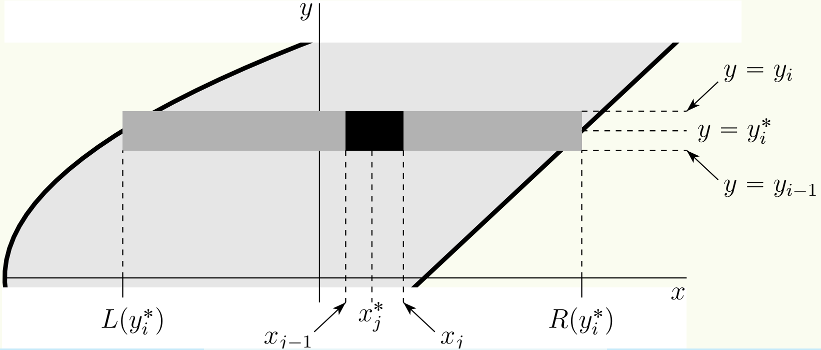


Horizontal Slice (Con'td)

- We approximate slice i by a rectangle whose **left** side is at $x = L(y_i^*)$ and whose **right** side is at $x = R(y_i^*)$.
- If we were computing the area of $\boldsymbol{R}$, we would now approximate the area of slice i by $\left[R(x_i^*) - L(x_i^*)\right]\Delta y$, which is the area of the rectangle with width $\left[R(x_i^*) - L(x_i^*)\right]$ and height Δy.

Horizontal Rectangle

- Subdivide slice number i into m tiny rectangles, each of height Δy and of width $\Delta x = \dfrac{1}{m}\left[R(y_i^*) - L(y_i^*)\right]$.
- For each $j = 1, 2, \ldots, m$, rectangle number j has x running over a very narrow range. We pick a number x_j^* somewhere in that range.



Riemann Sum Approximation

+ On rectangle number j in slice number i, we approximate the density by $f\left(x_j^*, y_i^*\right)$, giving us that the mass of rectangle number j in slice number i is approximately $f\left(x_j^*, y_i^*\right) \Delta x \, \Delta y$.
+ So the Riemann sum approximation of the mass of (horizontal) slice number i is

$$\text{Mass of slice } i \approx \sum_{j=1}^{m} f\left(x_j^*, y_i^*\right) \Delta x \, \Delta y.$$

+ By taking the limit as $m \to \infty$, we convert the Riemann sum into a **definite integral**:

$$\text{Mass of slice } i \approx \Delta y \int_{L(y_i^*)}^{R(y_i^*)} f\left(x, y_i^*\right) \, \mathrm{d}x = F(y_i^*) \, \Delta y.$$

where $F(y) = \int_{L(y)}^{R(y)} f\left(x, y\right) \, \mathrm{d}x$.

Exact Mass

- Observe that, as x has been integrated out, $F(y)$ is a function of y only, not of x and y.
- Finally taking the limit as $n \to \infty$ (i.e. taking the limit as the slice width goes to zero), we get

$$\text{Mass} = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta y \int_{L(y_i^*)}^{R(y_i^*)} f\left(x, y_i^*\right)\, \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} F(y_i^*)\, \Delta y.$$

- Now $\sum_{i=1}^{n} F(y_i^*)\, \Delta y$ is a Riemann sum approximation to the integral $\int_c^d F(y)\, \mathrm{d}y$. So

$$\text{Mass} = \int_c^d F(y)\, \mathrm{d}y = \int_c^d \left[\int_{L(y)}^{R(y)} f\left(x, y\right)\, \mathrm{d}x\right] \mathrm{d}y.$$

Notation for Left and Right

+ Iterated integrals

$$\iint_{\boldsymbol{R}} f(x, y)\ \mathrm{d}x\ \mathrm{d}y = \int_c^d \left[\int_{L(y)}^{R(y)} f(x, y)\ \mathrm{d}x\right] \mathrm{d}y$$

$$= \int_c^d \int_{L(y)}^{R(y)} f(x, y)\ \mathrm{d}x\ \mathrm{d}y$$

$$= \int_c^d \mathrm{d}y \int_{L(y)}^{R(y)} \mathrm{d}x\, f(x, y).$$

Evaluation of $\int_c^d \int_{L(y)}^{R(y)} f(x,y)\ \mathrm{d}x\ \mathrm{d}y$

- First evaluate the inside integral $\int_{L(y)}^{R(y)} f(x,y)\ \mathrm{d}x$ using the inside limits of integration. The result of the inside integral is a function of y only. Call it $F(y)$.
- Then evaluate the outside integral $\int_c^d F(y)\ \mathrm{d}y$, whose **integrand** is the answer to the inside integral.

Evaluation of $\int_c^d \mathrm{d}y \int_{L(y)}^{R(y)} \mathrm{d}x\, f(x, y)$

- First evaluate the inside integral $\int_{L(y)}^{R(y)} \mathrm{d}x\, f(x, y)$ using the limits of integration that are directly beside the $\mathrm{d}x$.
- Again, the $\mathrm{d}x$ is written directly beside $\int_{L(y)}^{R(y)}$ to make it clear that the limits of integration $L(y)$ and $R(y)$ are for the x-integral.
- In the past you probably wrote this integral as $\int_{L(y)}^{R(y)} f(x, y)\, \mathrm{d}x$. The result of the inside integral is again a function of y only. Call it $F(y)$.
- Then evaluate the outside integral $\int_c^d \mathrm{d}y\, F(y)$, whose **integrand** is the answer to the inside integral and whose limits of integration are directly beside the $\mathrm{d}y$.

Summary

Theorem 3.1.

a *If* $\boldsymbol{R} = \left\{ (x, y) \,\middle|\, a \le x \le b,\ B(x) \le y \le T(x) \right\}$ *with* $B(x)$ *and* $T(x)$ *being continuous, and if the mass density in* $\boldsymbol{R}$ *is* $f(x, y)$*, then the mass of* $\boldsymbol{R}$ *is*

$$\int_a^b \left[\int_{B(x)}^{T(x)} f(x, y)\, \mathrm{d}y \right] \mathrm{d}x = \int_a^b \int_{B(x)}^{T(x)} f(x, y)\, \mathrm{d}y\, \mathrm{d}x = \int_a^b \mathrm{d}x \int_{B(x)}^{T(x)} \mathrm{d}y f(x, y).$$

b *If* $\boldsymbol{R} = \left\{ (x, y) \,\middle|\, c \le y \le d,\ L(y) \le x \le R(y) \right\}$ *with* $L(y)$ *and* $R(y)$ *being continuous, and if the mass density in* $\boldsymbol{R}$ *is* $f(x, y)$*, then the mass of* $\boldsymbol{R}$ *is*

$$\int_c^d \left[\int_{L(y)}^{R(y)} f(x, y)\, \mathrm{d}x \right] \mathrm{d}y = \int_c^d \int_{L(y)}^{R(y)} f(x, y)\, \mathrm{d}x\, \mathrm{d}y = \int_c^d \mathrm{d}y \int_{L(y)}^{R(y)} \mathrm{d}x f(x, y).$$

Fubini's Theorem

+ The integrals of Theorem 3.1 are often denoted

$$\iint_{\boldsymbol{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{or} \qquad \iint_{\boldsymbol{R}} f(x,y)\,\mathrm{d}A$$

  The symbol $\mathrm{d}A$ represents the area of an "**infinitesimal**" piece of $\boldsymbol{R}$.

+ Implicit in Slide 25 is the statement that, if

$$\{\,(x,y)\mid a\le x\le b,\ B(x)\le y\le T(x)\,\}=\{\,(x,y)\mid c\le y\le d,\ L(y)\le x\le R(y)\,\}$$

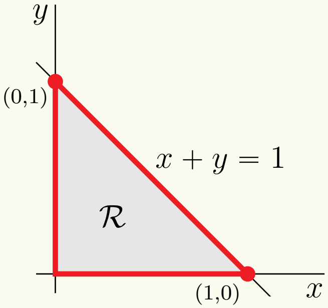
  and if $f(x,y)$ is **continuous**, then

$$\int_a^b\int_{B(x)}^{T(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x=\int_c^d\int_{L(y)}^{R(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

+ This is called **Fubini's theorem**, named after the Italian mathematician Guido Fubini (1879–1943).

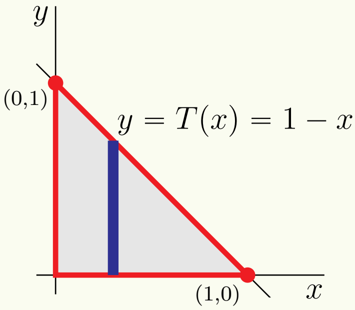
A Simple Example

- Let $\boldsymbol{R}$ be the triangular region above the x-axis, to the right of the y-axis and to the left of the line $x + y = 1$.
- Find the mass of $\boldsymbol{R}$ if it has density $f(x, y) = y$.



Solution Using Vertical Strips

- Note that the leftmost points in $\boldsymbol{R}$ have $x = 0$ and the rightmost point in $\boldsymbol{R}$ has $x = 1$.



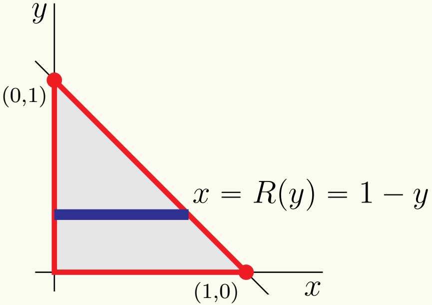
- For each fixed x between 0 and 1, the point (x, y) in $\boldsymbol{R}$ with the smallest y has $y = 0$ and the point (x, y) in $\boldsymbol{R}$ with the largest y has $y = 1 - x$.

Solution Using Vertical Strips (Cont'd)

- Thus
$$R = \left\{(x, y)\middle|0 = a \le x \le b = 1,\ 0 = B(x) \le y \le T(x) = 1 - x\right\}$$
- By part (a) of Theorem 3.1
$$\text{Mass} = \int_a^b \mathrm{d}x \int_{B(x)}^{T(x)} \mathrm{d}y\, f\big(x, y\big) = \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, y.$$
- Now the inside integral is
$$\int_0^{1-x} y\ \mathrm{d}y = \left[\frac{y^2}{2}\right]_0^{1-x} = \frac{1}{2}(1-x)^2,$$
so that the
$$\text{Mass} = \int_0^1 \mathrm{d}x\, \frac{(1-x)^2}{2} = \left[-\frac{(1-x)^3}{6}\right]_0^1 = \boxed{\frac{1}{6}}.$$

Solution Using Horizontal Strips

- Note the lowest points in $\boldsymbol{R}$ have $y = 0$ and the topmost point in $\boldsymbol{R}$ has $y = 1$.



- For each fixed y between 0 and 1, the point (x, y) in $\boldsymbol{R}$ with the smallest x has $x = 0$ and the point (x, y) in $\boldsymbol{R}$ with the largest x has $x = 1 - y$.

Solution Using Horizontal Strips (Cont'd)

- Thus
$$R = \{(x,y) \big| 0 = c \le y \le d = 1,\ 0 = L(y) \le x \le R(y) = 1 - y\}.$$
- By part (b) of Theorem 3.1
$$\text{Mass} = \int_c^d \mathrm{d}y \int_{L(y)}^{R(y)} \mathrm{d}x\, f(x,y) = \int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}x\, y.$$
- Now the inside integral is
$$\int_0^{1-y} y \ \mathrm{d}x = [xy]_0^{1-y} = y - y^2,$$
since the y integral treats x as a constant, so
$$\text{Mass} = \int_0^1 \mathrm{d}y\ [y - y^2] = \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Arithmetic of Integration

Theorem 4.1 (Arithmetic of Integration).

Let A, B, C be real numbers. Under the hypotheses of Theorem 3.1,

(a) $$\iint_{\boldsymbol{R}} (f(x,y) + g(x,y))\, \mathrm{d}x\, \mathrm{d}y = \iint_{\boldsymbol{R}} f(x,y)\, \mathrm{d}x\, \mathrm{d}y + \iint_{\boldsymbol{R}} g(x,y)\, \mathrm{d}x\, \mathrm{d}y$$

(b) $$\iint_{\boldsymbol{R}} (f(x,y) - g(x,y))\, \mathrm{d}x\, \mathrm{d}y = \iint_{\boldsymbol{R}} f(x,y)\, \mathrm{d}x\, \mathrm{d}y - \iint_{\boldsymbol{R}} g(x,y)\, \mathrm{d}x\, \mathrm{d}y$$

(c) $$\iint_{\boldsymbol{R}} C f(x,y)\, \mathrm{d}x\, \mathrm{d}y = C \iint_{\boldsymbol{R}} f(x,y)\, \mathrm{d}x\, \mathrm{d}y$$

Arithmetic of Integration (Con'td)

Theorem 4.1 ((Con'td)).

Combining these three rules we have

(d) $$\iint_{\boldsymbol{R}} (Af(x,y) + Bg(x,y))\, \mathrm{d}x\, \mathrm{d}y = A \iint_{\boldsymbol{R}} f(x,y)\, \mathrm{d}x\, \mathrm{d}y + B \iint_{\boldsymbol{R}} g(x,y)\, \mathrm{d}x\, \mathrm{d}y$$

(e) $$\iint_{\boldsymbol{R}} \mathrm{d}x\, \mathrm{d}y = \textit{Area}(\boldsymbol{R})$$

(f) $$\iint_{\boldsymbol{R}} f(x,y)\, \mathrm{d}x\, \mathrm{d}y = \iint_{\boldsymbol{R}_1} f(x,y)\, \mathrm{d}x\, \mathrm{d}y + \iint_{\boldsymbol{R}_2} f(x,y)\, \mathrm{d}x\, \mathrm{d}y$$

if the two regions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ do not intersect.

Special Case

Theorem 4.2.

If the domain of integration

$$\boldsymbol{R} = \{(x,y)|a \le x \le b,\ c \le y \le d\}$$

is a rectangle and the integrand is the product $f(x,y) = g(x)h(y)$*, then*

$$\iint_{\boldsymbol{R}} f(x,y)\ \mathrm{d}x\,\mathrm{d}y = \left[\int_a^b \mathrm{d}x\ g(x)\right]\ \left[\int_c^d \mathrm{d}y\ h(y)\right].$$

Inequalities for Integrals (1)

Theorem 4.3 (Inequalities for Integrals).

Under the assumptions of Theorem 3.1,

a. *If* $f(x,y) \geq 0$ *for all* (x,y) *in* $\boldsymbol{R}$*, then*

$$\iint_{\boldsymbol{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \geq 0.$$

b. *If there are constants* m *and* M *such that* $m \leq f(x,y) \leq M$ *for all* (x,y) *in* $\boldsymbol{R}$*, then*

$$m\,\textit{Area}(\boldsymbol{R}) \leq \iint_{\boldsymbol{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq M\,\textit{Area}(\boldsymbol{R}).$$

Inequalities for Integrals (2)

Theorem 4.3 (Con'td).

c *If* $f(x, y) \leq g(x, y)$ *for all* (x, y) *in* $\boldsymbol{R}$*, then*

$$\iint_{\boldsymbol{R}} f(x, y)\ \mathrm{d}x\,\mathrm{d}y \leq \iint_{\boldsymbol{R}} g(x, y)\ \mathrm{d}x\,\mathrm{d}y.$$

d *We have*

$$\left|\iint_{\boldsymbol{R}} f(x, y)\ \mathrm{d}x\,\mathrm{d}y\right| \leq \iint_{\boldsymbol{R}} |f(x, y)|\ \mathrm{d}x\,\mathrm{d}y.$$

Example 1

- Evaluate $\iint\limits_D 42y^2 - 12x \, \mathrm{d}x \, \mathrm{d}y$ where $D = \left\{(x,y) \,|0 \leq x \leq 4, (x-2)^2 \leq y \leq 6\right\}$.
- Define $I := \iint\limits_D 42y^2 - 12x \, \mathrm{d}x \, \mathrm{d}y = \int_0^4 \int_{(x-2)^2}^6 42y^2 - 12x \, \mathrm{d}y \, \mathrm{d}x$.
- Hence,

$$I = \int_0^4 \int_{(x-2)^2}^6 42y^2 - 12x \, dy \, dx = \int_0^4 \left(14y^3 - 12xy\right)\Big|_{(x-2)^2}^6 \, dx$$

$$= \int_0^4 3024 - 72x - 14(x-2)^6 + 12x(x-2)^2 \, dx$$

$$= \cdots .$$

Example 2

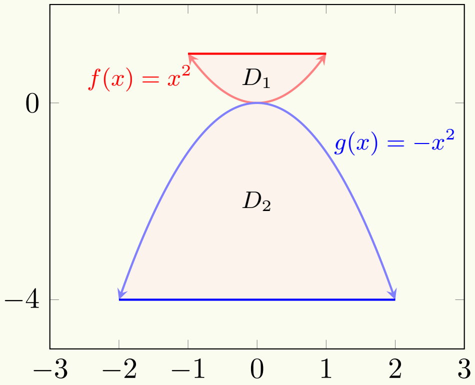
- Evaluate $I := \iint_D 2yx^2 + 9y^3 \, \mathrm{d}x \, \mathrm{d}y$ where D is the region bounded by $y = \frac{2}{3}x$ and $y = 2\sqrt{x}$.
- $$\begin{matrix} 0 \le x \le 9 \\ \frac{2}{3}x \le y \le 2\sqrt{x} \end{matrix} \quad \text{or} \quad \begin{matrix} 0 \le y \le 6 \\ \frac{1}{4}y^2 \le x \le \frac{3}{2}y \end{matrix}$$

$$I = \int_0^9 \int_{\frac{2}{3}x}^{2\sqrt{x}} 2yx^2 + 9y^3 \, \mathrm{d}y \, \mathrm{d}x = \int_0^9 \left(y^2x^2 + \frac{9}{4}y^4 \right) \bigg|_{\frac{2}{3}x}^{2\sqrt{x}} \mathrm{d}x$$

$$= \int_0^9 4x\left(x^2\right) + \frac{9}{4}\left(16x^2\right) - \left[\frac{4}{9}x^2\left(x^2\right) + \frac{9}{4}\left(\frac{16}{81}x^4\right) \right] \mathrm{d}x$$

$$= \int_0^9 36x^2 + 4x^3 - \frac{8}{9}x^4 \, \mathrm{d}x = 12x^3 + x^4 - \frac{8}{45}x^5 \bigg|_0^9 = \boxed{\frac{24057}{5}}.$$

Example 3 Question



- What is $\iint\limits_D 3 - 6xy \, \mathrm{d}x \, \mathrm{d}y$, where D is the region shown on the left?

Example 3 Answer

- We write $$\iint\limits_D 3-6xy\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{D_1} 3-6xy\,\mathrm{d}x\,\mathrm{d}y + \iint\limits_{D_2} 3-6xy\,\mathrm{d}x\,\mathrm{d}y.$$

- The two regions are

$$D_1 \quad -1 \le x \le 1 \quad x^2 \le y \le 1,$$

$$D_2 \quad -2 \le x \le 2 \quad -4 \le y \le -x^2.$$

- Therefore, $$\iint\limits_D 3-6xy\,\mathrm{d}x\,\mathrm{d}y = \int_{-1}^{1}\int_{x^2}^{1} 3-6xy\,\mathrm{d}y\,\mathrm{d}x + \int_{-2}^{2}\int_{-4}^{-x^2} 3-6xy\,\mathrm{d}y\,\mathrm{d}x\ .$$

Example 3 Answer (Cont'd)

- It follows that

$$\iint_D 3 - 6xy \,\mathrm{d}x\,\mathrm{d}y = \int_{-1}^{1} \left(3y - 3xy^2\right)\Big|_{x^2}^{1} \,\mathrm{d}x + \int_{-2}^{2} \left(3y - 3xy^2\right)\Big|_{-4}^{-x^2} \,\mathrm{d}x$$

$$= \int_{-1}^{1} 3x^5 - 3x^2 - 3x + 3 \,\mathrm{d}x + \int_{-2}^{2} 12 + 48x - 3x^2 - 3x^5 \,\mathrm{d}x.$$

- The answer is

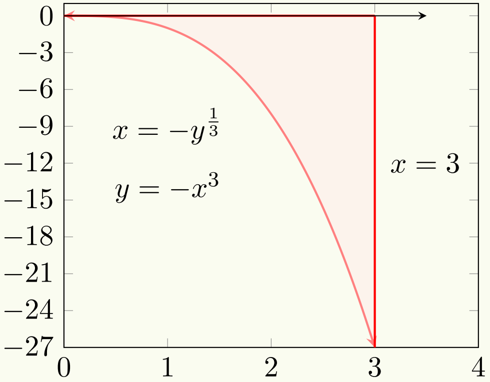
$$\left(\frac{1}{2}x^6 - x^3 - \frac{3}{2}x^2 + 3x\right)\Big|_{-1}^{1} + \left(12x + 24x^2 - x^3 - \frac{1}{2}x^6\right)\Big|_{-2}^{2}$$

$$= 4 + 32$$

$$= \boxed{36}.$$

Example 4

- Evaluate $\displaystyle\iint_D \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\,\mathrm{d}x\,\mathrm{d}y$ where D is the region bounded by $x=-y^{\frac{1}{3}}, x=3$ and the x-axis.



Example 4 Answer

- $$\begin{aligned} 0 \le x \le 3 \\ -x^3 \le y \le 0 \end{aligned}$$ or $$\begin{aligned} -27 \le y \le 0 \\ -y^{\frac{1}{3}} \le x \le 3 \end{aligned}$$
- Thus,

$$\begin{aligned}
\iint\limits_D \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^3 \int_{-x^3}^0 \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\,\mathrm{d}y\,\mathrm{d}x = \int_0^3 \left(\frac{\frac{3}{2}y^{\frac{2}{3}}}{\left(x^3+1\right)}\right)\Bigg|_{-x^3}^{0}\,\mathrm{d}x \\
&= \int_0^3 -\frac{\frac{3}{2}\left(-x^3\right)^{\frac{2}{3}}}{\left(x^3+1\right)}\,\mathrm{d}x = -\int_0^3 \frac{\frac{3}{2}x^2}{\left(x^3+1\right)}\,\mathrm{d}x \\
&= -\int_0^3 \frac{\frac{1}{2}}{\left(x^3+1\right)}\,\mathrm{d}\left(x^3\right) = -\frac{1}{2}\ln\left|x^3+1\right|\Big|_0^3 \\
&= -\frac{\ln(28)}{2}.
\end{aligned}$$

Example

- Evaluate $I = \int_0^\infty \left(\frac{1 - e^{-x}}{x} \right)^2 \, \mathrm{d}x$.

- Solution: Reverse the integration

$$I = \int_0^\infty \left(\frac{-e^{-xy}}{x} \Bigg|_{y=0}^{y=1} \right)^2 \mathrm{d}x = \int_0^\infty \left(\int_0^1 e^{-xy} \, \mathrm{d}y \right)^2 \mathrm{d}x = \int_0^\infty \int_0^1 e^{-xy} \, \mathrm{d}y \int_0^1 e^{-xz} \, \mathrm{d}z \, \mathrm{d}x$$

$$= \int_0^1 \int_0^1 \int_0^\infty e^{-x(y+z)} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

- It is easy to integrate $\int_0^\infty e^{-x(y+z)} \, \mathrm{d}x$, resulting in

$$\int_0^\infty e^{-x(y+z)} \, \mathrm{d}x = \frac{-e^{-x(y+z)}}{y+z} \Bigg|_{x=0}^{x=\infty} = \frac{1}{y+z}.$$

- Therefore.

$$I = \int_0^1 \int_0^1 \frac{1}{y+z} \, \mathrm{d}y \, \mathrm{d}z = \int_0^1 \ln(y+z) \Big|_{y=0}^{y=1} \mathrm{d}z = \int_0^1 \left(\ln(z+1) - \ln z \right) \, \mathrm{d}z$$

- By integration by parts, we know that $\int \ln x \, \mathrm{d}x = x \ln x - x + C$.

- The answer is

$$\begin{aligned} I &= \left((z+1)\ln(z+1) - (\cancel{z}+1) - z\ln z + \cancel{z} \right) \Big|_{z=0}^{z=1} \\ &= \boxed{2\ln 2}. \end{aligned}$$

Keywords