Section 6

Application of Calculus and Critical Thinking: Cosmology 101

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Cautionary Quote

Yet in another way, calculus is fundamentally naive, almost childish in its optimism. Experience teaches us that change can be sudden, discontinuous, and wrenching. Calculus draws its power by refusing to see that. It insists on a world without accidents, where one thing leads logically to another. Give me the initial conditions and the law of motion, and with calculus I can predict the future—or better yet, reconstruct the past.

— Steven Strogatz


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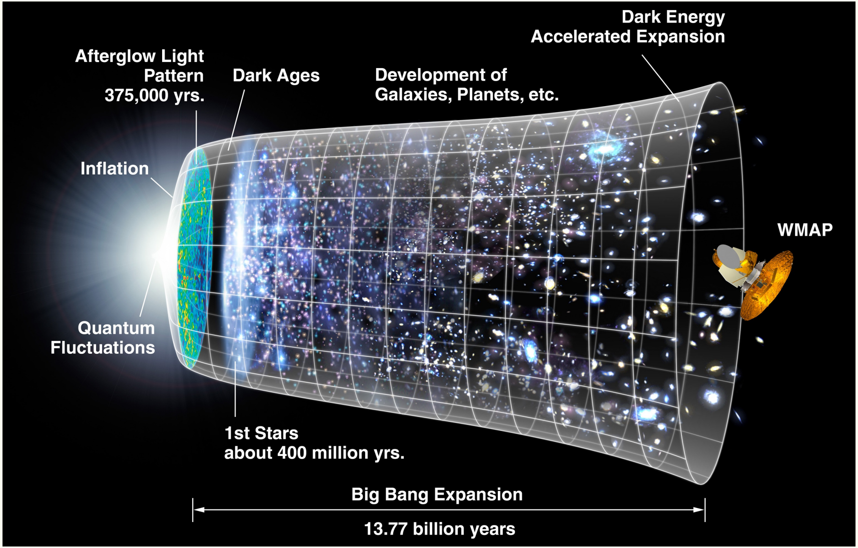
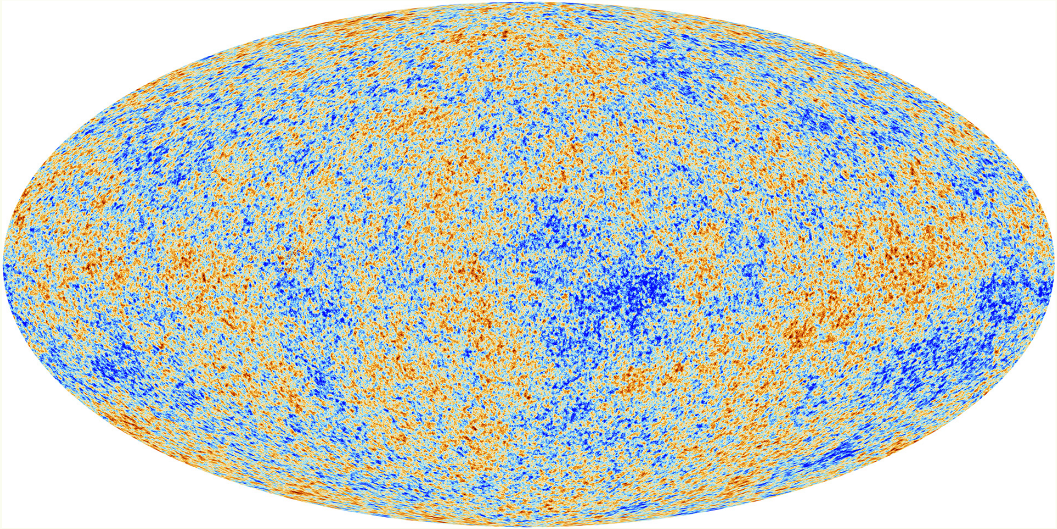
Inflationary Big Bang Paradigm



Planck's Cosmic Microwave Background

Credit: European Space Agency's Planck **CMB**.



Einstein's General Theory of Relativity

- **Einstein Theory of General Relativity**
  Spacetime $G_{\mu\nu}$ tells **matter** $T_{\mu\nu}$ how to move, and matter tells **spacetime** how to curve:
  $$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \qquad (1)$$
  where G is **Newton's constant of gravitation** and c is the speed of light in **vacuum**.
- Since $G = 6.67430 \times 10^{-11}$ m^3 kg^{-1} s^{-2} and $c = 299{,}792{,}458$ m/s, the value of **Einstein's gravitational constant** κ is
  $$\kappa := \frac{8\pi G}{c^4} = 2.077 \times 10^{-43}\,\text{s}^2\,\text{kg}^{-1}\,\text{m}^{-1}\ .$$
- How can we solve **Einstein's field equations** (1)?

Simplifying Assumptions

1. **Cosmological principle A** : **isotropic**; i.e., no particular preferred direction

2. Cosmological principle B : **homogeneous**, uniform; no big void in the universe

3. **Perfect fluid** : the content in the **universe** is like a fluid that is perfectly determined by the (rest frame) **mass-energy density** ρ and isotropic **pressure** P.

These 3 assumptions are challenged by observational evidence.

If the assumptions are invalid, can the model built upon them be correct?

Conservation of Energy

- A system of **fluid** has **internal energy** U due to the **kinetic energy** + **potential energy** of its molecules.
- The **first law of thermodynamics** states that

$$\mathrm{d}Q = \mathrm{d}U + \mathrm{d}W,$$

  where
  - $\mathrm{d}Q$ is the differential change of (external) **heat added** to the system
  - $\mathrm{d}W$ is the differential change of **work done** by the system
  - $\mathrm{d}U$ is the differential change in **internal energy** of the system
- The universe is an isolated system, so $\mathrm{d}Q = 0$, i.e., **adiabatic**.
- Therefore the content in the **universe** is modeled as a system of **adiabatic perfect fluid**.

Universe as a Sphere

- A system of **volume** V that grows in size: $\mathrm{d}W = P\,\mathrm{d}V$, where P is the **pressure**.
- Universe is a **sphere**, so $V = \frac{4}{3}\pi a^3$ of **radius** a. Since ρ is the **density**, we have

$$U = \frac{4}{3}\pi a^3 \rho.$$

- Upon **differentiation** of U with respect to time t, by **product rule** and **chain rule**, we obtain

$$\frac{\mathrm{d}U}{\mathrm{d}t} = 4\pi a^2 \rho \frac{\mathrm{d}a}{\mathrm{d}t} + \frac{4}{3}\pi a^3 \frac{\mathrm{d}\rho}{\mathrm{d}t}. \quad (2)$$

- The rate of change of volume is given by

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi a^2 \frac{\mathrm{d}a}{\mathrm{d}t}. \quad (3)$$

Adiabatic Perfect Fluid Equation

- Define $\dot{a} := \frac{\mathrm{d}a}{\mathrm{d}t}$ and $\dot{\rho} := \frac{\mathrm{d}\rho}{\mathrm{d}t}$.
- Under the assumption of $\mathrm{d}Q = 0$, i.e., $\mathrm{d}U = -P\,\mathrm{d}V$, we obtain from (2) and (3)

$$4\pi a^2(\rho + P)\dot{a} + \frac{4}{3}\pi a^3\dot{\rho} = 0.$$

- Assume that $a(t) \neq 0$ at all time, we obtain the **fluid equation**

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0. \tag{4}$$

- The **fluid equation** describes how the **density** of **universe** changes with time.

Friedmann's Metric and Stress-Energy Tensor

- The Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + a(t)^2\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right],$$

where $a(t)$ is the **scale factor**.

- The symbol k can be either -1, 0, or 1, which is interpreted as describing, respectively, an universe that is either open, flat, or closed.

- Assuming that the universe is filled with a perfect fluid, then the energy-matter tensor is

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right)u^\mu u^\nu + Pg^{\mu\nu},$$

where $u^\mu = (1, 0, 0, 0)$ in the co-moving frame, leading to $T_0^0 = -\rho c^2$, and $T_j^i = P\delta_j^i$.

First Friedmann Equation

- The time component G_{00} of Einstein tensor (1)

$$G_{00} = 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3kc^2}{a^2}.$$

- Plugging it to Einstein's equation for the time component, i.e.,

$$G_{00} = \frac{8\pi G}{c^4} T_{00} = \frac{8\pi G}{c^2}\rho,$$

we obtain

$$3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3kc^2}{a^2} = \frac{8\pi G}{c^2}\rho \qquad \Longrightarrow \qquad \boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho - \frac{kc^2}{a^2}} \tag{5}$$

Second Friedmann Equation

- We compute the trace of spatial components:

$$G_{ii} = -a^2 \left(2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} \right)$$

- Plugging it to Einstein's equation for the spatial components, i.e.,

$$G_{ii} = \frac{8\pi G}{c^4} T_{ii} = \frac{8\pi G}{c^2} P a^2,$$

we obtain

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2} P \qquad (6)$$

Second Friedmann Equation (continued)

- Using the first Friedmann equation (5) to substitute $(\dot{a}/a)^2$, we obtain

$$2\frac{\ddot{a}}{a} + \frac{8\pi G}{3c^2}\rho - \frac{kc^2}{a^2} + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2}P.$$

- The expression simplifies to the second Friedmann equation

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\rho + \frac{3P}{c^2}\right)}$$

- The symbol $\dot{a} := \frac{\mathrm{d}a}{\mathrm{d}t}$ is the **speed** of expansion.

- The symbol $\ddot{a} := \frac{\mathrm{d}^2 a}{\mathrm{d}t^2}$ is the **acceleration**of expansion.

Simplest Cosmology

- When $P = 0$, the **fluid equation** (4) becomes $\dot{\rho} + 3\rho\frac{\dot{a}}{a} = 0$.
- Assuming that $a \neq 0$, after multiplying both sides by a^3, we apply the **product rule** to obtain

$$a^3\dot{\rho} + 3\rho\dot{a}a^2 = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}(a^3\rho) = 0.$$

- So $a^3\rho$ is a constant. We write it as $a_0^3\rho_0$, where a_0 is the current **size** of the **universe**, and ρ_0 is the current **density**.
- Hence the density is a function of $a(t)$:

$$\rho = \left(\frac{a_0}{a}\right)^3 \rho_0. \tag{7}$$

Ordinary Differential Equation

- We set $k = 0$.
- Since ρ is given by (7), the first Friedmann equation (5) then becomes a simple first-order **ordinary differential equation**:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho = \frac{8\pi G}{3c^2}\left(\frac{a_0}{a}\right)^3 \rho_0. \tag{8}$$

- We obtain

$$\boxed{\dot{a} := \frac{\mathrm{d}a}{\mathrm{d}t} = \sqrt{\frac{8\pi G \rho_0 a_0^3}{3c^2}} \frac{1}{\sqrt{a}}.}$$

Simplest Solution

- Applying **calculus**, it can be solved easily as follows:

$$\sqrt{a}\ \mathrm{d}a = \sqrt{\frac{8\pi G\rho_0 a_0^3}{3c^2}}\,\mathrm{d}t \quad \Longrightarrow \quad \int \sqrt{a}\ \mathrm{d}a = \sqrt{\frac{8\pi G\rho_0 a_0^3}{3c^2}} \int \mathrm{d}t$$

- Thus,

$$\frac{2}{3}a^{\frac{3}{2}} = \sqrt{\frac{8\pi G\rho_0 a_0^3}{3c^2}}\,t + C, \qquad (\spadesuit)$$

where C is the **constant of integration**.

- How should we determine C?

Universe is Expanding

- Let time $t = 0$ be today. Let the **size** of today's **universe** be the standard, and thus we let $a_0 \equiv a(0) = 1$.
- Let $\rho_0 = \dfrac{3c^2 H_0^2}{8\pi G}$, where H_0 is called the **Hubble constant**.
- The coefficient on the right-hand side of (♠) becomes H_0. So we have

$$\frac{2}{3} a^{\frac{3}{2}} = H_0 t + C.$$

- Upon rewriting, we find that $a(t)$ is a simple **power function** of time t:

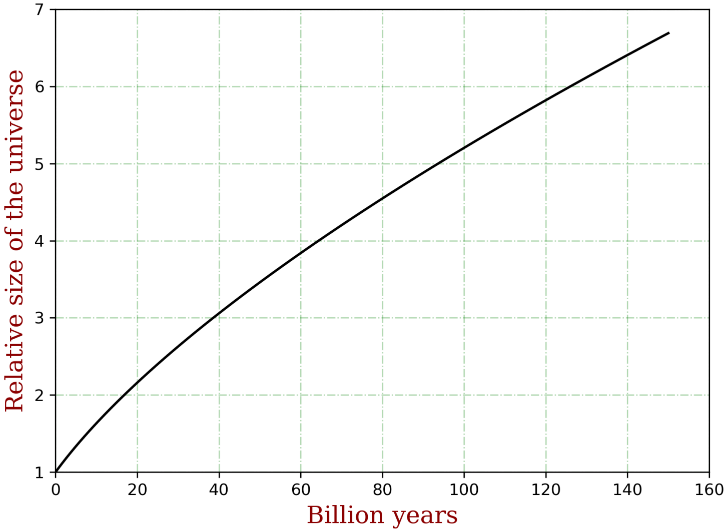
$$a(t) = \left(\frac{3H_0}{2}\right)^{\frac{2}{3}} t^{\frac{2}{3}} + C. \tag{9}$$

Determining the Constant of Integration

- The **initial condition** is that at time 0, $a(0) = a_0 = 1$, we get $C = 1$.
- So the **universe** is expanding as a simple function of time $t \geqq 0$:

$$a(t) = \left(\frac{3H_0}{2}t\right)^{\frac{2}{3}} + 1. \tag{10}$$

Forecast of the Simplest Friedmann Cosmology



"Age" of the Flat Universe (1 of 2)

- Assuming that we can unwind the **universe** back to the time T when its **size** is infinitesimally small.
- In (9), let $a(T)=0$, and we obtain $C=-\left(\frac{3H_0}{2}T\right)^{\frac{2}{3}}$.
- Instead of (10) for $t>0$, we have $a(t)=\left(\frac{3H_0}{2}\right)^{\frac{2}{3}}\left(t^{\frac{2}{3}}-T^{\frac{2}{3}}\right)$.
- Let $a(0)=1$ for today, and we obtain,

$$1=\left(\frac{3H_0}{2}(-T)\right)^{\frac{2}{3}}.$$

- Therefore, $\boxed{T=-\frac{2}{3H_0}}$.

"Age" of the Flat Universe (2 of 2)

- According to **Hubble's law** on Wikipedia, we can take 2022-02-08 estimate of $73.4(\mathrm{km/s})/\mathrm{Mpc}$.
- According to **UnitConverts.net**, 1 Mpc is equal to $3.08567758128 \times 10^{19}\mathrm{km}$.
- Therefore,

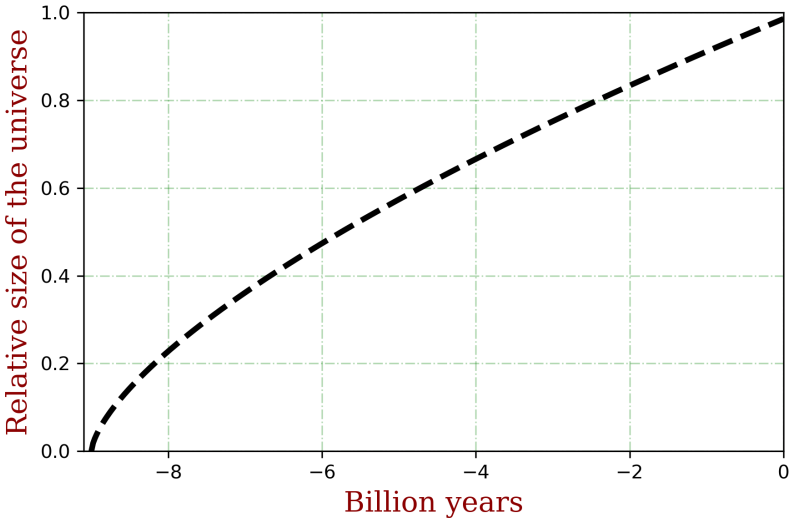
$$\frac{1}{H_0} = \frac{3.08567758128 \times 10^{19}}{73.4} = 4.20392 \times 10^{17}\mathrm{s} = \frac{4.20392 \times 10^{17}}{24 \cdot 60 \cdot 60 \cdot 365.25} = 13.32 \times 10^{9}\mathrm{years}.$$

- It follows that the "age" of the flat universe is

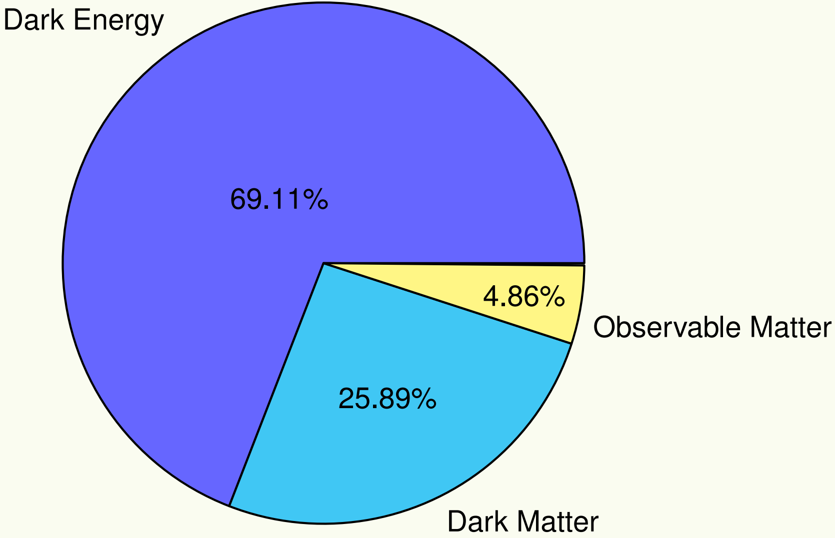
$$T = -\frac{2}{3H_0} = -\text{ 8.88 billion years.}$$

- Given the negative sign, if extrapolation backward was valid, the **Big Bang** would have happened 8.88 billion years ago in the simplest Friedmann cosmology.

Backcast of the Simplest Friedmann Cosmology

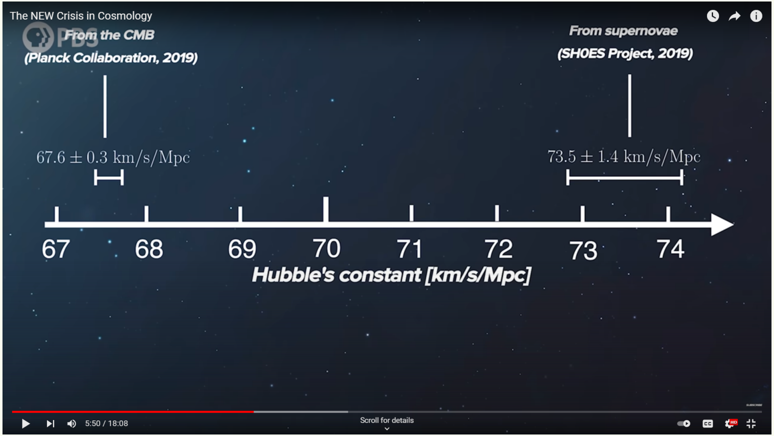


How much do we know about our universe?



Crisis in Cosmology

- Watch **the NEW Crisis in Cosmology**.
- Different methods to measure the **Hubble constant** should give the same result.

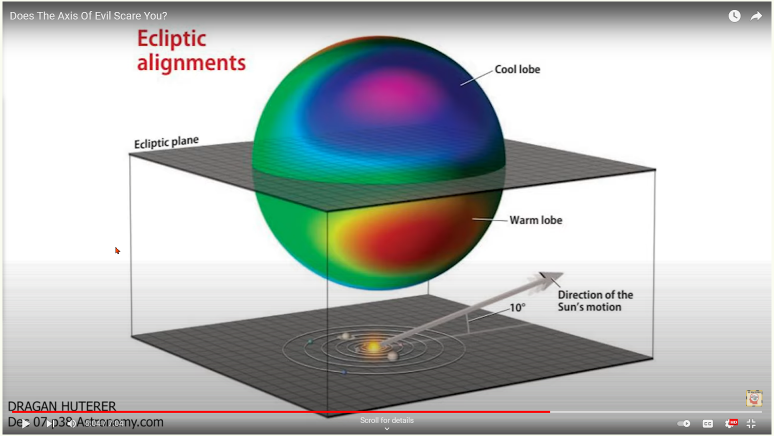


Cosmological Constant Problem

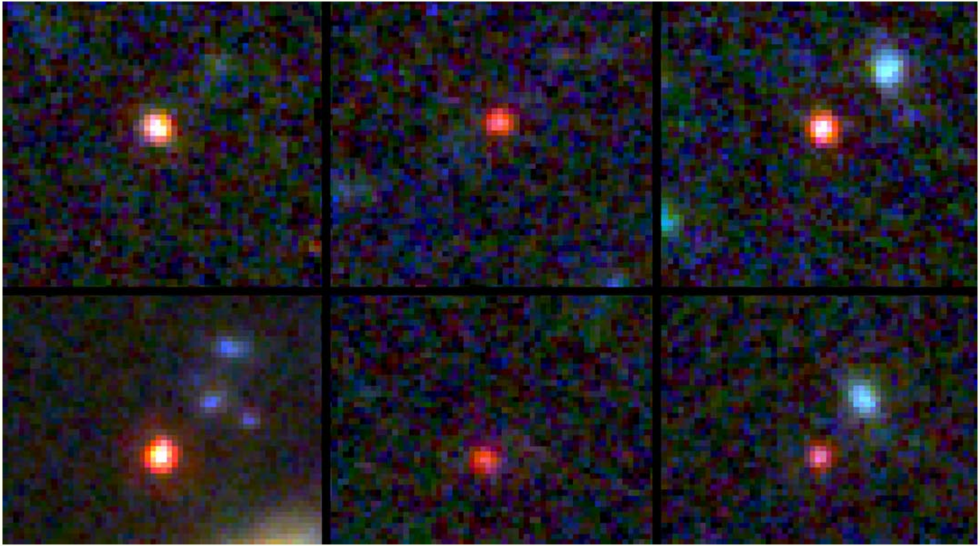
- The Λ in the ΛCDM **consensus model** is the **energy density** of empty space.
- Theoretically, by quantum field theory, there should be about 10^{112} erg for every cubic centimeter of empty space.
- But the actual value obtained by observation is 10^{-8} erg per cubic centimeter.
- Theoretical prediction is 10^{120} times the energy density of empty space that is actually observed.
- Check out **the Best and Worst Prediction in Science**.

Cosmic Axis of Evil

- See **Does The Axis Of Evil Scare You?**.
- The axis of CMB universe aligns exactly with the plane of our solar system.



Universe Breaker Galaxies



More Big Problems in Big Bang

- How did time begin?
- Matter and anti-matter are formed by vacuum through "quantum fluctuation". Why does the universe contain only matter? Where had the anti-matter gone?
- Where had the monopoles gone?
- Is dark matter real?
- Is dark energy real?
- Massive galaxies older than the universe?
- How did the moon form?

Keywords