Section 6
Integration Techniques

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Example 1: Integration by Parts I

Use **integration by parts** to evaluate $\int_0^\infty x\, e^{-x}\, dx$.

Solution:

$$\int u\, dv = uv - \int v\, du$$

$$\int \underbrace{x}_{u}\, \underbrace{e^{-x}\, dx}_{dv} = \underbrace{x}_{u}\, \underbrace{(-e^{-x})}_{v} - \int \underbrace{-e^{-x}}_{v}\, \underbrace{dx}_{du}$$

$$\int x\, e^{-x}\, dx = -x\, e^{-x} - e^{-x} + C$$

Example 1 (Cont'd)

$$\int_0^\infty x\,e^{-x}\,dx = \left(-x\,e^{-x} - e^{-x}\right)\Big|_0^\infty$$

$$= \lim_{x\to\infty}\left(-x\,e^{-x} - e^{-x}\right) - \left(-0\,e^0 - e^0\right)$$

$$= -\left(\lim_{x\to\infty}\frac{x}{e^x}\right) - \left(\lim_{x\to\infty} e^{-x}\right) - (0-1)$$

$$= 0 - 0 + 1 = \boxed{1}.$$

What if?

- What would happen if we let $u = e^{-x}$ and $dv = x\,dx$?
- *Answer:* In this case $du = -e^{-x}\,dx$ and $v = \int dv = \int x\,dx = \frac{1}{2}x^2$, so that

$$\begin{aligned}\int x\,e^{-x}\,dx &= \int \underbrace{e^{-x}}_{u}\,\underbrace{x\,dx}_{dv} \\ &= \underbrace{e^{-x}}_{u}\,\underbrace{\frac{1}{2}x^2}_{v} - \int \underbrace{\frac{1}{2}x^2}_{v}\,\underbrace{(-e^{-x})\,dx}_{du} \\ &= \frac{1}{2}x^2\,e^{-x} + \frac{1}{2}\int x^2\,e^{-x}\,dx\end{aligned}$$

- It leads us to the wrong direction—a more difficult integral than the original.

What's going on?

- Evaluate $\displaystyle\int \frac{dx}{x}$ using **integration by parts**.
- Let $u = \dfrac{1}{x}$ *and* $dv = dx$, *so that* $du = -\dfrac{dx}{x^2}$ *and* $v = x$,
- Then

$$\int u\,dv = uv - \int v\,du$$

$$\int \frac{dx}{x} = \left(\frac{1}{x}\right)\cdot x - \int x \cdot \left(-\frac{dx}{x^2}\right)$$

$$\int \frac{dx}{x} = 1 + \int \frac{dx}{x}$$

$$0 = 1 \quad ?$$

General Case of Failure

- How does this "contradiction" occur?
- Integrating by parts the following integral $I = \int \frac{f'(x)}{f(x)} dx$.
- Let $u = \frac{1}{f(x)}$ and $dv = f'(x)dx$. Thus $du = -\frac{f'(x)}{(f(x))^2} dx$ and $v = f(x)$.

$$
\begin{aligned}
I = \int \frac{f'(x)}{f(x)} dx &= \frac{1}{f(x)} f(x) - \int \left(-\frac{f'(x)}{(f(x))^2} \right) f(x) dx \\
&= 1 + \int \frac{f'(x)}{f(x)} dx.
\end{aligned}
$$

- We then get $I = 1 + I$.
- There is no contradiction because the **constants of integration** are different on both sides. We should not use **integration by parts** in the first place.

Example 2: More than One Integration by Parts

- Evaluate $\int x^2 e^{-x} \, dx$.

Solution:

- Choose $u = x^2$ and so $dv = e^{-x} \, dx$. Then $du = 2x \, dx$ and $v = \int e^{-x} \, dv = -e^{-x}$.
- Integrate by parts with the formula $\int u \, dv = uv - \int v \, du$.

$$\int x^2 e^{-x} \, dx = x^2 \cdot (-e^{-x}) - \int -e^{-x} \cdot 2x \, dx$$

$$= -x^2 e^{-x} + 2 \int x \, e^{-x} \, dx \quad \text{(integrate by parts again)}$$

$$\int x^2 e^{-x} \, dx = -x^2 e^{-x} + 2 \left(-x \, e^{-x} - e^{-x}\right) + C \quad \text{(by Example 1)}$$

$$= -x^2 e^{-x} - 2x \, e^{-x} - 2 \, e^{-x} + C.$$

Integration by Parts n Rounds

- In general, if n rounds of **integration by parts** were needed, with u_i and v_i representing the u and v, respectively, for round $i = 1, 2, \ldots, n$, then

$$\begin{aligned}\int u_1\,dv_1 &= u_1v_1 - \int v_1\,du_1 = u_1v_1 - \int u_2\,dv_2 \\ &= u_1v_1 - \left(u_2v_2 - \int v_2\,du_2\right) = u_1v_1 - u_2v_2 + \int u_3\,dv_3 \\ &= u_1v_1 - u_2v_2 + \left(u_3v_3 - \int u_4\,dv_4\right) \\ &= u_1v_1 - u_2v_2 + u_3v_3 - \left(u_4v_4 - \int u_5\,dv_5\right) = \cdots \\ &= u_1v_1 - u_2v_2 + u_3v_3 - u_4v_4 + u_5v_5 - \cdots \int u_n\,dv_n.\end{aligned}$$

- The last integral $\int u_n\,dv_n$ is one you could presumably integrate easily.

Tabular Method

- Let the arrows indicate multiplication. Then

$$
\begin{array}{cc}
u & dv \\
\hline
u_1 & dv_1 \\
u_2 \searrow & v_1 \quad (+) \longrightarrow + u_1 v_1 \\
u_3 \searrow & v_2 \quad (-) \longrightarrow - u_2 v_2 \\
u_3 \searrow & v_3 \quad (+) \longrightarrow + u_3 v_3 \\
u_4 \searrow & v_4 \quad (-) \longrightarrow - u_4 v_4 \\
\vdots & \vdots \quad \vdots
\end{array}
$$

- The idea is to differentiate down the u column and integrate down the dv column.
- This **tabular method** allows the **integration by parts** to be performed efficiently and correctly.

Example of Tabular Method

- The **tabular method** on the integral from the previous example looks like this:

	u		dv			
	x^2	↘	$e^{-x}\,dx$			
	$2x$	↘	$-e^{-x}$	(+)	→	$+\,(x^2)\,(-e^{-x})$
	2	↘	e^{-x}	(−)	→	$-\,(2x)\,(e^{-x})$
STOP →	0		$-e^{-x}$	(+)	→	$+\,(2)\,(-e^{-x})$

- The integral is the sum of the products, and agrees with the result:

$$\int x^2\,e^{-x}\,dx = +\,(x^2)\,(-e^{-x}) - (2x)\,(e^{-x}) + (2)\,(-e^{-x}) + C$$

$$= -x^2\,e^{-x} - 2x\,e^{-x} - 2\,e^{-x} + C$$

Example 3

Evaluate $\int x^3 e^{-x}\, dx$.

Solution: Use the **tabular method** with $u = x^3$ and $dv = e^{-x}\, dx$ as follows:

	u	dv		
	x^3	$e^{-x}\, dx$		
	$3x^2$	$-e^{-x}$	(+)	$\longrightarrow + (x^3)(-e^{-x})$
	$6x$	e^{-x}	(−)	$\longrightarrow - (3x^2)(e^{-x})$
	6	$-e^{-x}$	(+)	$\longrightarrow + (6x)(-e^{-x})$
STOP $\longrightarrow$	0	e^{-x}	(−)	$\longrightarrow - (6)(e^{-x})$

$$\int x^3 e^{-x}\, dx = -x^3 e^{-x} - 3x^2 e^{-x} - 6x\, e^{-x} - 6\, e^{-x} + C.$$

Gamma Function

- In general, we have **Gamma function**:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \quad \text{for all } n > 0.$$

- It is a generalization of **factorial**.

$$\boxed{\Gamma(n) = (n-1)!} \quad \text{for all positive whole numbers larger than 0.}$$

- From Example 3:

$$\Gamma(4) = \int_0^\infty x^3 e^{-x}\, dx = \left(-x^3 e^{-x} - 3x^2 e^{-x} - 6x\, e^{-x} - 6\, e^{-x}\right)\Big|_0^\infty$$

$$= (-0-0-0-0) - (-0-0-0-6) = 6.$$

Example 4: Summary

Evaluate $\int_0^1 x^3 \sqrt{1-x^2}\, dx$.

Solution: Use the method of **integration by parts**.

- We have $x^3 \sqrt{1-x^2} = x^2 \cdot x\sqrt{1-x^2}$. It is easy to integrate $x\sqrt{1-x^2}$.
- So let $u = x^2$ and $dv = x\sqrt{1-x^2}\, dx$.
- Then $du = 2x\, dx$ and $v = \int dv = \int x\sqrt{1-x^2}\, dx = -\frac{1}{3}(1-x^2)^{3/2}$. Thus,

$$\int_0^1 x^3 \sqrt{1-x^2}\, dx = -\frac{x^2}{3}(1-x^2)^{3/2}\Big|_0^1 + \int_0^1 \frac{2x}{3}(1-x^2)^{3/2}\, dx$$

$$= (0-0) + \left(\frac{-2}{15}(1-x^2)^{5/2}\Big|_0^1\right) = 0 + \frac{2}{15} = \boxed{\frac{2}{15}}.$$

Preparation: $I = \int \sec(x)\,dx = \ln|\tan(x) + \sec(x)| + C$

- We write $\int \sec(x)dx = \int \frac{1}{\cos(x)}dx = \int \frac{\cos(x)}{\cos^2(x)}dx = \int \frac{\cos(x)}{1-\sin^2(x)}dx$.
- Then use the **substitution** $y = \sin(x)$, $dy = \cos(x)dx$ and then use **partial fractions**.

$$I = \int \frac{1}{1-y^2}dy = \int \frac{1}{(1+y)(1-y)}dy = \frac{1}{2}\int \frac{1}{1+y} + \frac{1}{1-y}dy$$
$$= \frac{1}{2}\int \frac{1}{y+1} - \frac{1}{y-1}dy = \frac{1}{2}\left(\ln|y+1| - \ln|y-1|\right) + C = \frac{1}{2}\ln\left|\frac{\sin(x)+1}{\sin(x)-1}\right| + C.$$

- After a bit of rewriting, we obtain

$$\sqrt{\left|\frac{\sin(x)+1}{\sin(x)-1}\right|} = \sqrt{\left|\frac{(\sin(x)+1)^2}{\sin^2(x)-1}\right|} = \sqrt{\left|\frac{(\sin(x)+1)^2}{-\cos^2(x)}\right|} = \left|\frac{\sin(x)+1}{\cos(x)}\right| = |\tan(x)+\sec(x)|.$$

Example 5

Evaluate $\int \sec^3 x \, dx$.

Solution: Let $u = \sec x$ and $dv = \sec^2 x \, dx$. Use the integration by parts.

- So $du = \sec x \; \tan x \, dx$ and $v = \int dv = \int \sec^2 x \, dx = \tan x$. Then

$$
\begin{aligned}
\int \sec^3 x \, dx &= \sec x \; \tan x - \int \sec x \; \tan^2 x \, dx \\
\int \sec^3 x \, dx &= \sec x \; \tan x - \int \sec x \, (\sec^2 x - 1) \, dx \\
\int \sec^3 x \, dx &= \sec x \; \tan x + \int \sec x \, dx - \int \sec^3 x \, dx \\
2 \int \sec^3 x \, dx &= \sec x \; \tan x + \ln |\sec x + \tan x| + C \\
\int \sec^3 x \, dx &= \frac{1}{2} (\sec x \; \tan x + \ln |\sec x + \tan x|) + C
\end{aligned}
$$

Product to Sum

$$\sin A \cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right) \tag{1}$$

$$\cos A \sin B = \frac{1}{2}\left(\sin(A+B) - \sin(A-B)\right) \tag{2}$$

$$\cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right) \tag{3}$$

$$\sin A \sin B = -\frac{1}{2}\left(\cos(A+B) - \cos(A-B)\right) \tag{4}$$

Example 6

Evaluate $\int \frac{1}{2} \sin x \, \sin 12x \, dx$.

Solution: Using the **product-to-sum formula** (4) with $A = x$ and $B = 12x$,

$$\begin{aligned} \sin A \sin B &= -\tfrac{1}{2}\left(\cos(A+B) - \cos(A-B)\right) \\ \sin x \sin 12x &= -\tfrac{1}{2}\left(\cos(x+12x) - \cos(x-12x)\right) \\ \sin x \sin 12x &= -\tfrac{1}{2}\left(\cos 13x - \cos 11x\right) \end{aligned}$$

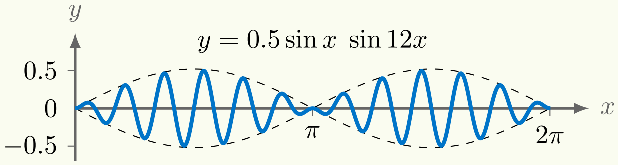
since $\cos(-11x) = \cos 11x$. Then

$$\begin{aligned} \int \frac{1}{2} \sin x \, \sin 12x \, dx &= -\frac{1}{4} \int (\cos 13x - \cos 11x) \, dx \\ &= -\frac{1}{52} \sin 13x + \frac{1}{44} \sin 11x + C. \end{aligned}$$

Discussion of Example 6

- Notice how the **product-to-sum formula** turned an integral of products of sines into integrals of individual cosines, which are easily integrated.

- The integrand is an example of a **modulated wave**, commonly used in electronic communications (e.g. radio broadcasting).



- The curves $y = \pm 0.5 \sin\, x$ (shown in dashed lines) form an **amplitude envelope** for the modulated wave.

Higher Power of Sine

- For the sine function raised to odd powers of the form $2n+1$ (for $n \geqq 1$), the trick is to replace $\sin^2 x$ by $1-\cos^2 x$.

$$\begin{aligned}\int \sin^{2n+1} x \, dx &= \int (\sin^2 x)^n \, \sin \, x \, dx \\ &= \int (1-\cos^2 x)^n \, \sin \, x \, dx \\ &= \int p(u) \, du.\end{aligned}$$

- The function $p(u)$ is a **polynomial** in the variable $u = \cos \, x$, and the remaining single $\sin \, x$ is now part of $du = -\sin \, x \, dx$.
- Then use the power formula to integrate that polynomial.

Example 7

Evaluate $\int \sin^3 x \, dx$.

Solution: Let $u = \cos x$ so that $du = -\sin x \, dx$:

$$\begin{aligned}
\int \sin^3 x \, dx &= \int (\sin^2 x) \, \sin x \, dx = \int (1 - \cos^2 x) \, \sin x \, dx \\
&= \int (1 - u^2) \, (-du) = \int (u^2 - 1) \, du \\
&= \frac{1}{3} u^3 - u + C \\
&= \frac{1}{3} \cos^3 x - \cos x + C.
\end{aligned}$$

Higher Power of Cosine

- In general $\int \sin^{2n+1} x\, dx$ will be a **polynomial** of degree $2n+1$ in terms of $\cos x$.
- Similarly, use $\cos^2 x = 1 - \sin^2 x$ to integrate odd powers of $\cos x$, with the substitution $u = \sin x$:

$$\begin{aligned}\int \cos^{2n+1} x\, dx &= \int (\cos^2 x)^n \cos x\, dx \\ &= \int \underbrace{(1-\sin^2 x)^n}_{p(u)} \underbrace{\cos x\, dx}_{du}.\end{aligned}$$

- Integrals of the form $\int \sin^m x \cos^n x\, dx$, where either m or n is odd, can be evaluated using the above trick for the function having the odd power.

Example 8

Evaluate $\int \sin^2 x \ \cos^3 x \, dx$.

Solution: Replace $\cos^2 x$ by $1 - \sin^2 x$, then let $u = \sin\, x$ so that $du = \cos\, x\, dx$:

$$\int \sin^2 x \ \cos^3 x \ dx = \int \sin^2 x \,(\cos^2 x) \ \cos\, x\, dx = \int \underbrace{\sin^2 x\,(1 - \sin^2 x)}_{p(u)} \ \underbrace{\cos\, x\, dx}_{du}$$

$$= \int (u^2 - u^4)\, du$$

$$= \frac{1}{3} u^3 - \frac{1}{5} u^5 + C$$

$$= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C.$$

Example 9: Even Power of Sine

Evaluate $\int \sin^4 x \, dx$.

Solution: Replace $\sin^2 x$ by $\dfrac{1 - \cos 2x}{2}$:

$$\begin{aligned}
\int \sin^4 x \, dx &= \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx = \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) \, dx \\
&= \frac{1}{4} \int \left(1 - 2 \cos 2x + \frac{1 + \cos 4x}{2}\right) dx \\
&= \frac{1}{8} \int (3 - 4 \cos 2x + \cos 4x) \, dx \\
&= \frac{3x}{8} - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C.
\end{aligned}$$

Even Power of Sine and Cosine

- For even powers of $\sin x$ or $\cos x$, replace $\sin^2 x$ or $\cos^2 x$ with either

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{or} \quad \cos^2 x = \frac{1 + \cos 2x}{2},$$

respectively, as often as necessary, then proceed as before if odd powers occur.

$\sec x$ **with** $\tan x$ (1 of 2)

- Similar methods can be used for integrals of the form $\int \sec^m x \ \tan^n x \, dx$ when either m is even or n is odd.

- For an even power $m = 2k + 2$, use $\sec^2 x = 1 + \tan^2 x$ for all but two of the m powers of $\sec\ x$, then use the substitution $u = \tan\ x$, so that $du = \sec^2 x \ dx$.

- This results in an integral of a **polynomial** $p(u)$ in terms of $u = \tan\ x$:

$$\begin{aligned}\int \sec^{2k+2} x \ \tan^n x \, dx &= \int (\sec^2 x)^k \sec^2 x \ \tan^n x \, dx \\ &= \int \underbrace{(1+\tan^2 x)^k \ \tan^n x}_{p(u)} \ \underbrace{\sec^2 x \, dx}_{du} .\end{aligned}$$

$\sec x$ with $\tan x$ (2 of 2)

- Likewise for an odd power $n = 2k+1$, use $\tan^2 x = \sec^2 x - 1$ for all but one of the n powers of $\tan x$, then use the **substitution** $u = \sec x$, so that $du = \sec x \tan x \, dx$.
- This results in an integral of a **polynomial** $p(u)$ in terms of $u = \sec x$:

$$\int \sec^m x \tan^{2k+1} x \, dx = \int \sec^{m-1} x \sec x \, (\tan^2 x)^k \tan x \, dx$$

$$= \int \underbrace{\sec^{m-1} x \, (\sec^2 x - 1)^k}_{p(u)} \underbrace{\sec x \tan x dx}_{du}$$

- Mimic the above procedure for integrals of the form $\int \csc^m x \cot^n x \, dx$ when either m is even or n is odd, using the identity $\csc^2 x = 1 + \cot^2 x$ in a similar manner.

Example 10

Evaluate $\int \sec^4 x \, \tan x \, dx$.

Solution: Use $\sec^2 x = 1 + \tan^2 x$ for one $\sec^2 x$ term, then substitute $u = \tan x$, so that $du = \sec^2 x \, dx$:

$$\begin{aligned}
\int \sec^4 x \, \tan x \, dx &= \int \sec^2 x \, \sec^2 x \, \tan x \, dx = \int (1 + \tan^2 x) \, \tan x \, \sec^2 x \, dx \\
&= \int (1 + u^2) \, u \, du = \int (u + u^3) \, du \\
&= \frac{1}{2} u^2 + \frac{1}{4} u^4 + C \\
&= \frac{1}{2} \tan^2 x + \frac{1}{4} \tan^4 x + C.
\end{aligned}$$

Example 11

- For some **trigonometric integrals** try putting everything in terms of sines and cosines.

- Evaluate $\displaystyle\int \frac{\cot^4 x}{\csc^5 x}\, dx$.

  Solution: Put $\cot\, x$ and $\csc\, x$ in terms of $\sin\, x$ and $\cos\, x$:

$$
\begin{aligned}
\int \frac{\cot^4 x}{\csc^5 x}\, dx &= \int \frac{\cos^4 x\ \sin^5 x}{\sin^4 x}\, dx \\
&= \int \cos^4 x\ \sin\, x\, dx \quad \text{(now let } u = \cos\, x,\, du = -\sin\, x\, dx\text{)} \\
&= -\int u^4\, dx = -\frac{1}{5}\cos^5 x + C.
\end{aligned}
$$

Example 12: Gamma Function

- It is easy to see that $F(t) := \int_0^\infty e^{-tx} dx = \frac{1}{t}$ for all $t > 0$.
- Differentiating $F(t)$ with respect to t leads to

$$F'(t) = -\int_0^\infty x e^{-tx} dx = -\frac{1}{t^2} \quad \Longrightarrow \quad \int_0^\infty x e^{-tx} dx = \frac{1}{t^2}$$

- Taking further derivatives yields

$$F^{(n)}(t) = \int_0^\infty x^n e^{-tx} dx = \frac{n!}{t^{n+1}}.$$

- When t is set equal to one, we obtain the **Gamma function**:

$$n! = \int_0^\infty x^n e^{-x} dx := \Gamma(n+1).$$

Example 13

- Compute $\int_0^1 \frac{x^2-1}{\ln(x)} dx$.
- Define $G(t) := \int_0^1 \frac{x^t-1}{\ln(x)} dx$. Then $G(2)$ is the answer we are seeking. Obviously $G(0)=0$.
- Differential $G(t)$ withe respect to t yields

$$G'(t) = \int_0^1 \frac{\partial}{\partial t}\left(\frac{x^t-1}{\ln(x)}\right) dx = \int_0^1 x^t dx = \frac{1}{t+1}.$$

- Therefore

$$G(2) = \int_0^2 G'(t)dt = \int_0^2 \frac{1}{t+1} dt = \ln(t+1)\Big|_0^2 = \ln(3).$$

Example 14: The Dirichlet Integral

- Evaluate $\int_0^\infty \frac{\sin(x)}{x}\,dx$ using **Feynman's technique**.
- Let $f(s) := \int_0^\infty e^{-sx}\frac{\sin(x)}{x}\,dx$, which is a function of s.
- Note that $f(0)$ is the integral we want to evaluate.
- $f'(s) = \int_0^\infty \frac{\partial}{\partial s} e^{-sx}\frac{\sin(x)}{x}\,dx = \int_0^\infty -e^{-sx}\sin(x)\,dx$.
- Performing the **integration by parts** with $u = \sin(x)$ and $dv = -e^{-sx}dx$, we obtain $du = \cos(x)dx$ and $\int dv = v = \int -e^{-sx}dx = \frac{1}{s}e^{-sx}$.
- Therefore, $f'(s) = \frac{1}{s}\sin(x)e^{-sx}\Big|_0^\infty - \frac{1}{s}\int_0^\infty \cos(x)e^{-sx}dx = -\frac{1}{s}\int_0^\infty \cos(x)e^{-sx}dx$.

Example 14: The Dirichlet Integral (con'td)

- We need to perform integration by parts one more time on $-\dfrac{1}{s}\displaystyle\int_0^\infty \cos(x)e^{-sx}dx$.

- We let $u = \cos x$, and hence $du = -\sin(x)dx$. And with $dv = -\dfrac{1}{s}e^{-sx}dx$, we obtain $\displaystyle\int dv = v = \int -\frac{1}{s}e^{-sx}dx = \frac{1}{s^2}e^{-sx}$.

- Therefore,

$$\begin{aligned} f'(s) &= -\int_0^\infty \sin(x)e^{-sx}dx = \frac{1}{s}\int_0^\infty \cos(x)e^{-sx}dx \\ &= \frac{1}{s^2}\cos(x)e^{-sx}\Big|_0^\infty + \frac{1}{s^2}\int_0^\infty \sin(x)e^{-sx}dx \\ &= -\frac{1}{s^2} + \frac{1}{s^2}\int_0^\infty \sin(x)e^{-sx}dx. \end{aligned}$$

Example 14: The Dirichlet Integral (con'td)

- Thus we obtain $f'(s) = -\frac{1}{s^2} - \frac{1}{s^2} f'(s)$.
- After a re-arrangement of terms, we obtain $f'(s) = -\frac{1}{1+s^2}$.
- Now, we perform an improper integration $f(s) = -\int \frac{1}{1+s^2} ds = -\arctan(s) + C$.
- We need to know what C is. For this purpose, we consider

$$\lim_{s\to\infty} f(s) = \lim_{s\to\infty} \int_0^{\infty} e^{-sx} \frac{\sin(x)}{x}\, dx = \int_0^{\infty} \lim_{s\to\infty} e^{-sx} \frac{\sin(x)}{x}\, dx = 0.$$

- On the other hand, $\lim_{s\to\infty} f(s) = \lim_{s\to\infty} -\arctan(s) + C = -\frac{\pi}{2} + C$.

Example 14: The Dirichlet Integral (con'td)

- Hence we have
$$C = \lim_{s\to\infty} \arctan(s) = \frac{\pi}{2}.$$

- Finally, we obtain
$$f(s) = \int_0^\infty e^{-sx}\frac{\sin(x)}{x}\,dx = -\arctan(s) + \frac{\pi}{2}.$$

- Since $\arctan(0) = 0$, it must be that
$$f(0) = \boxed{\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}}.$$

Chi-Square Probability Density Function

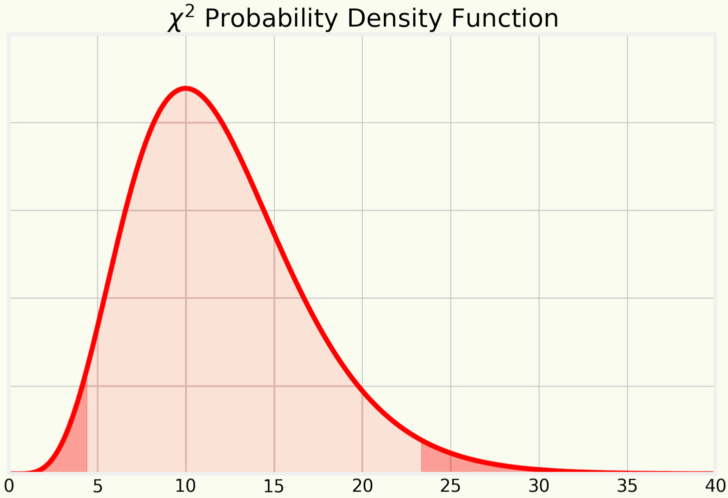
- The **analytical form** of the **chi-square probability density function** (pdf) with v **degrees of freedom** is, for $x \geqq 0$,

$$f(x;v) = \frac{\mathrm{e}^{-\frac{x}{2}}\, x^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\,\Gamma\left(\frac{v}{2}\right)} = C\mathrm{e}^{-\frac{x}{2}}\, x^{\frac{v}{2}-1},$$

where C is the constant term $\dfrac{1}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)}$.

- So $f(x;v)$ is a product of **exponential function** and a **power function**.
- Slide 37 provides a plot of the **chi-square pdf** $f(x;12)$ with 12 degrees of freedom

Plot of Chi-Square PDF



Moment Generating Function

- First we compute the **moment generating function**

$$M_X(t) = \mathbb{E}\left(\mathrm{e}^{tX}\right) = C\int_0^\infty \mathrm{e}^{tx}\mathrm{e}^{-\frac{x}{2}}x^{\frac{v}{2}-1}dx = C\int_0^\infty \mathrm{e}^{-\left(\frac{1}{2}-t\right)x}x^{\frac{v}{2}-1}dx.$$

- We perform a **change of variable** $y = \left(\dfrac{1}{2} - t\right)x$. Hence, $dy = \left(\dfrac{1}{2} - t\right)dx$, equivalently, $x = \dfrac{2}{1-2t}y$ and $dx = \dfrac{2}{1-2t}dy$, and we get

$$M_X(t) = C\int_0^\infty \mathrm{e}^{-y}\left(\frac{2}{1-2t}y\right)^{\frac{v}{2}-1}\frac{2}{1-2t}dy = C\int_0^\infty \mathrm{e}^{-y}\left(\frac{2}{1-2t}\right)^{\frac{v}{2}-1}y^{\frac{v}{2}-1}\frac{2}{1-2t}dy$$

$$= C\left(\frac{2}{1-2t}\right)^{\frac{v}{2}}\int_0^\infty \mathrm{e}^{-y}y^{\frac{v}{2}-1}dy.$$

Gamma function

- By definition, the integral is the **Gamma function** $\Gamma\left(\frac{\upsilon}{2}\right)$. Consequently,

$$M_X(t) = \frac{1}{2^{\frac{\upsilon}{2}}\Gamma\left(\frac{\upsilon}{2}\right)}\left(\frac{2}{1-2t}\right)^{\frac{\upsilon}{2}}\Gamma\left(\frac{\upsilon}{2}\right) = \left(\frac{1}{1-2t}\right)^{\frac{\upsilon}{2}}. \tag{5}$$

- Next, we differentiate $M_X(t)$ with respect to t:

$$M_X'(t) = C\int_0^{\infty} x\,\mathrm{e}^{tx}\mathrm{e}^{-\frac{x}{2}}x^{\frac{\upsilon}{2}-1}dx. \tag{6}$$

It follows from (5) that

$$M_X'(t) = -\frac{\upsilon}{2}(-2)\left(\frac{1}{1-2t}\right)^{\frac{\upsilon}{2}+1} = \upsilon\left(\frac{1}{1-2t}\right)^{\frac{\upsilon}{2}+1}. \tag{7}$$

Mean of Chi-Square Random Variable

- Interestingly, the **expected value** of X can be obtained as follows:

$$M'_X(0) = C\int_0^\infty x\,\mathrm{e}^{-\frac{x}{2}} x^{\frac{\upsilon}{2}-1} dx = \mathbb{E}(X).$$

- Substituting 0 for t in (7), we obtain the mean of **chi-square random variable**:

$$\mathbb{E}(X) = M'_X(0) = \upsilon.$$

- The mean of X is its **number of degrees of freedom**.

Variance of Chi-Square Random Variable

- Next, we differentiate the **moment generating function** (5) with respect to t twice to obtain

$$M_X''(t) = C\int_0^\infty x^2\,\mathrm{e}^{tx}\mathrm{e}^{-\frac{x}{2}}x^{\frac{v}{2}-1}dx = v\,(v+2)\left(\frac{1}{1-2t}\right)^{\frac{v}{2}+2},$$

which is the **expected value** of X^2 when $t=0$.

- In other words,

$$\mathbb{E}\left(X^2\right) = M_X''(0) = v^2 + 2v.$$

- As $\mathbb{V}(X) = \mathbb{E}\left(X^2\right) - (\mathbb{E}(X))^2$, the **variance** of **chi-square random variable** with v **degrees of freedom** is

$$\mathbb{V}(X) = M_X''(0) - \left(M_X'(0)\right)^2 = v^2 + 2v - v^2 = 2v.$$

Keywords