Week 5
Application of Calculus:
Portfolio Optimization

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Inspiring Quote

A good portfolio is more than a long list of good stocks and bonds. It is a balanced whole, providing the investor with protections and opportunities with respect to a wide range of contingencies.

— Harry Markowitz

Learning Outcomes

- Define and describe what a vector derivative is and how it works.
- Obtain a quick glimpse into partial differentiation.
- Analyze the component details of matrix calculus.
- Formulate a Lagrangian for constrained optimization with Lagrange multipliers.
- Derive the first-order conditions by applying the matrix calculus.
- Solve the first-order conditions using matrix algebra.
- Discuss the efficient frontier.

Definition

- Let $\boldsymbol{x}$ be a column k-vector. Consider the function

$$g(\boldsymbol{x}) = g(x_1, x_2, \ldots, x_k) : \Re^k \longrightarrow \Re.$$

- The **vector derivative** is defined as

$$\frac{\partial}{\partial \boldsymbol{x}} g(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial}{\partial x_1} g(\boldsymbol{x}) \\ \frac{\partial}{\partial x_2} g(\boldsymbol{x}) \\ \vdots \\ \frac{\partial}{\partial x_k} g(\boldsymbol{x}) \end{bmatrix} \quad \text{and} \quad \frac{\partial}{\partial \boldsymbol{x}^\top} g(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial}{\partial x_1} g(\boldsymbol{x}) & \frac{\partial}{\partial x_2} g(\boldsymbol{x}) & \cdots & \frac{\partial}{\partial x_k} g(\boldsymbol{x}) \end{bmatrix}.$$

Basic Properties

- For constant vector $\boldsymbol{a}$ and matrix $\boldsymbol{A}$,

$$\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{a}^\top \boldsymbol{x}) = \frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{x}^\top \boldsymbol{a}) = \boldsymbol{a}, \qquad \frac{\partial}{\partial \boldsymbol{x}^\top}(\boldsymbol{a}^\top \boldsymbol{x}) = \boldsymbol{a}^\top$$

$$\frac{\partial}{\partial \boldsymbol{x}^\top}(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{A}$$

$$\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x}) = (\boldsymbol{A} + \boldsymbol{A}^\top)\boldsymbol{x}$$

$$\frac{\partial^2}{\partial \boldsymbol{x}\partial \boldsymbol{x}^\top}(\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x}) = \boldsymbol{A} + \boldsymbol{A}^\top$$

Illustration

- $f(x_1, x_2) := a_1 x_1 + a_2 x_2 = \boldsymbol{a}^\top \boldsymbol{x}$
- $\begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_2, x_2) \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \boldsymbol{A}\boldsymbol{x}.$

$$\frac{\partial}{\partial \boldsymbol{x}^\top}(\boldsymbol{A}\boldsymbol{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \boldsymbol{A}.$$

- $g(x_1, x_2) := \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} = a_{11}x_1^2 + a_{22}x_2^2 + a_{12}x_1x_2 + a_{21}x_1x_2$

$$\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x}) = \begin{bmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2a_{11}x_1 + a_{12}x_2 + a_{21}x_2 \\ 2a_{22}x_2 + a_{12}x_1 + a_{21}x_1 \end{bmatrix} = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

An Interesting Theorem of Matrix Calculation

Theorem 2.1 (Derivative of Trace).

Given two matrices $\boldsymbol{X}_{p\times q}$ *and* $\boldsymbol{A}_{q\times p}$. *Then*

$$\frac{\partial \mathrm{Tr}(\boldsymbol{X}\boldsymbol{A})}{\partial \boldsymbol{X}} = \boldsymbol{A}^\top.$$

- The product of $\boldsymbol{X}\boldsymbol{A}$ is

$$\boldsymbol{X}\boldsymbol{A} = \begin{bmatrix} x_{11} & \cdots & x_{1q} \\ \vdots & \ddots & \vdots \\ x_{p1} & \cdots & x_{pq} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{q1} & \cdots & a_{qp} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^q x_{1i}a_{i1} & \cdots & \sum_{i=1}^q x_{1i}a_{ip} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^q x_{pi}a_{i1} & \cdots & \sum_{i=1}^q x_{pi}a_{ip} \end{bmatrix}$$

- The **trace** of $\boldsymbol{X}\boldsymbol{A}$ is the sum of the **diagonal elements**.

$$\mathrm{Tr}(\boldsymbol{X}\boldsymbol{A}) = \sum_{i=1}^q x_{1i}a_{i1} + \cdots + \sum_{i=1}^q x_{pi}a_{ip} = \sum_{i=1}^q \sum_{j=1}^p x_{ji}a_{ij}$$

An Interesting Theorem of Matrix Calculation (Cont'd)

- Differentiate a function $f(\boldsymbol{X})$ with respect to $\boldsymbol{X}$ is

$$\frac{\partial f(\boldsymbol{X})}{\partial \boldsymbol{X}} = \begin{bmatrix} \frac{\partial f(\boldsymbol{X})}{\partial x_{11}} & \cdots & \frac{\partial f(\boldsymbol{X})}{\partial x_{1q}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f(\boldsymbol{X})}{\partial x_{p1}} & \cdots & \frac{\partial f(\boldsymbol{X})}{\partial x_{pq}} \end{bmatrix}$$

- Since $f(\boldsymbol{X}) = \sum_{i=1}^{q}\sum_{j=1}^{p} x_{ji}a_{ij}$, computation shows that

$$\frac{\partial \mathrm{Tr}(\boldsymbol{X}\boldsymbol{A})}{\partial \boldsymbol{X}} = \begin{bmatrix} \frac{\partial(\sum_{i=1}^{q}\sum_{j=1}^{p} x_{ji}a_{ij})}{\partial x_{11}} & \cdots & \frac{\partial(\sum_{i=1}^{q}\sum_{j=1}^{p} x_{ji}a_{ij})}{\partial x_{1q}} \\ \vdots & \ddots & \vdots \\ \frac{\partial(\sum_{i=1}^{q}\sum_{j=1}^{p} x_{ji}a_{ij})}{\partial x_{p1}} & \cdots & \frac{\partial(\sum_{i=1}^{q}\sum_{j=1}^{p} x_{ji}a_{ij})}{\partial x_{pq}} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{q1} \\ \vdots & \ddots & \vdots \\ a_{1p} & \cdots & a_{qp} \end{bmatrix} = \boldsymbol{A}^\top$$

□

Investment

- For each dollar invested, a fraction w_i is invested in **asset** i. It must be that

$$\sum_{i=1}^{n} w_i = 1.$$

- The **weights** are arranged as a n-vector $\boldsymbol{w}$.
- The **portfolio**'s **expected return** and **expected variance** are, respectively,

$$\mu_p := \mathbb{E}(r_p) = \sum_{i=1}^{n} w_i \mathbb{E}(r_i) = \sum_{i=1}^{n} w_i \, \mu_i = \boldsymbol{w}^\top \boldsymbol{\mu};$$

$$\sigma_p^2 := \mathbb{V}\left(r_p\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} = \boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}.$$

Numerical Illustration

- Suppose there are two assets. $\mu_1 = 5\%$ and $\mu_2 = 8\%$ **per annum**.
- The **covariance** is a 2 by 2 matrix.

$$\mathbf{\Sigma} := \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 0.0625 & -0.01 \\ -0.01 & 0.16 \end{bmatrix}$$

- Given that the variance of **asset** 1 is 0.0625, its **volatility** is $\sqrt{0.0625} = 25\%$ per annum.
- The volatility of asset 2 is ______________________.
- The **portfolio**'s **expected return** and **expected variance** are, respectively,

$$\begin{aligned} \mu_p &= 0.05w_1 + 0.08w_2 \\ \sigma_p^2 &= 0.0625w_1^2 - 0.01w_1w_2 - 0.01w_2w_1 + 0.16w_2^2 \end{aligned}$$

Optimization with Constraints

- Minimize half the **portfolio variance** under two **constrains**:

$$\sum_{i=1}^{n} w_i \mathbb{E}(r_i) = \mathbb{E}(r_p).$$

$$\sum_{i=1}^{n} w_i = 1.$$

- **Constrained optimization** with **Lagrange multipliers** λ and ψ:

$$\min_{w_1, w_2, \ldots, w_n} L = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} - \lambda \left(\sum_{i=1}^{n} w_i \, \mu_i - \mu_p \right) - \psi \left(\sum_{i=1}^{n} w_i - 1 \right)$$

In Matrix Form

- The **Lagrangian** L is

$$L = \frac{1}{2}\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w} - \lambda\left(\boldsymbol{w}^\top \boldsymbol{\mu} - \mu_p\right) - \psi(\boldsymbol{w}^\top \mathbf{1} - 1).$$

- Note that $\boldsymbol{\Sigma}$ is a **symmetric matrix**.
- The **first-order conditions** (**FOC**) with respect to $\boldsymbol{w}$, λ, and ψ are

$$\begin{aligned} \boldsymbol{\Sigma}\boldsymbol{w} - \lambda\boldsymbol{\mu} - \psi\mathbf{1} &= \mathbf{0} \\ \boldsymbol{w}^\top \boldsymbol{\mu} &= \mu_p \\ \boldsymbol{w}^\top \mathbf{1} &= 1 \end{aligned}$$

Solution of First FOC

- The first FOC gives the solution for the **weight vector**:

$$\boldsymbol{w}^* = \boldsymbol{\Sigma}^{-1}\big(\lambda\boldsymbol{\mu} + \psi\mathbf{1}\big).$$

- But what are the values of the **Lagrange multipliers** λ and ψ?
- To solve for λ and ψ, substitute the **optimal weight vector** above into the last two FOC's,

$$\begin{aligned} \boldsymbol{\mu}^\top\boldsymbol{w}^* &= \boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\big(\lambda\boldsymbol{\mu} + \psi\mathbf{1}\big) = \mu_p \\ \mathbf{1}^\top\boldsymbol{w}^* &= \mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\big(\lambda\boldsymbol{\mu} + \psi\mathbf{1}\big) = 1 \end{aligned}$$

Solution of Second and Third FOCs

- Let

$$a := \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$$
$$b := \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}$$
$$c := \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}$$

- The last two FOCs can be written as

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \lambda \\ \psi \end{bmatrix} = \begin{bmatrix} \mu_p \\ 1 \end{bmatrix}.$$

- Solving these two linear equations, we obtain

$$\lambda = \frac{c\mu_p - b}{ac - b^2}, \qquad \psi = \frac{a - b\mu_p}{ac - b^2}.$$

Optimal Weight Vector and Portfolio Variance

- The **optimal weight** is then solved as

$$\boldsymbol{w}^* = \frac{\boldsymbol{\Sigma}^{-1}\Big((c\mu_p - b)\boldsymbol{\mu} + (a - b\mu_p)\mathbf{1}\Big)}{ac - b^2}.$$

- The **portfolio variance** is a quadratic function of the **expected portfolio return** μ_p:

$$\sigma_P^2 := \mathbb{V}(r_p) = {\boldsymbol{w}^*}^\top \boldsymbol{\Sigma} \boldsymbol{w}^* = \frac{c\mu_p^2 - 2b\mu_p + a}{ac - b^2}$$
$$= \frac{c}{ac - b^2}\mu_p^2 - \frac{2b}{ac - b^2}\mu_p + \frac{a}{ac - b^2}.$$

Global Minimum Variance Portfolio

- The **global minimum variance portfolio** is obtained by minimizing $\mathbb{V}(r_p)$ with respect to μ_p.

$$\frac{d\mathbb{V}(r_p)}{d\mu_p} = \frac{2c}{ac - b^2}\mu_p - \frac{2b}{ac - b^2}.$$

- The results of the **first-order conditions** are

$$\mu_\star = \frac{b}{c}, \qquad \sigma^2_\star = \frac{1}{c}, \qquad \boldsymbol{w}_\star = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{c}.$$

Example

- For the two **assets**, compute the **inverse matrix**

$$\mathbf{\Sigma}^{-1} = \begin{bmatrix} 0.0625 & -0.01 \\ -0.01 & 0.16 \end{bmatrix}^{-1}$$

$$= \frac{1}{0.0625 \times 0.16 - (-0.01) \times (-0.01)} \begin{bmatrix} 0.16 & 0.01 \\ 0.01 & 0.0625 \end{bmatrix}$$

$$= \begin{bmatrix} 16.16 & 1.01 \\ 1.01 & 6.31 \end{bmatrix}$$

Three Scalars

- So the three **scalars** a, b, and c are

$$
\begin{aligned}
a &= \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \\
&= \begin{bmatrix} 0.05 & 0.08 \end{bmatrix} \begin{bmatrix} 16.16 & 1.01 \\ 1.01 & 6.31 \end{bmatrix} \begin{bmatrix} 0.05 \\ 0.08 \end{bmatrix} \\
&= 0.088864; \\
b &= \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} \\
&= \begin{bmatrix} 0.05 & 0.08 \end{bmatrix} \begin{bmatrix} 16.16 & 1.01 \\ 1.01 & 6.31 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= 1.4441 \\
c &= \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} \\
&= 24.49.
\end{aligned}
$$

Optimal Mean and Variance

- The **global minimum variance portfolio**

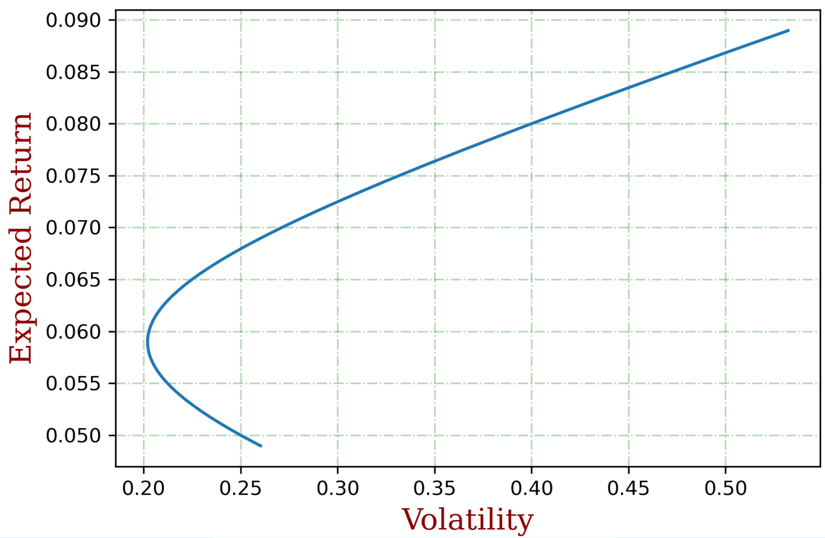
$$\mu_\star = \frac{b}{c} = 0.0590, \quad \sigma^2_\star = \frac{1}{c} = 0.0408.$$

- So the minimum **volatility** is $\sigma_\star = 20.21\%$.
- The **weight vector** for $w_\star$ for the global minimum variance portfolio is

$$\boldsymbol{w}_\star = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{c} = \frac{1}{24.49}\begin{bmatrix} 16.16 & 1.01 \\ 1.01 & 6.31 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 70.10\% \\ 29.90\% \end{bmatrix}.$$

- So invest 70.1% of your money in Asset 1 and 29.9% in Asset 2 to earn 5.90% **expected return** with a **volatility** of 20.21%.

Efficient Frontier



Keywords