Lesson 5
Calculus of Variation

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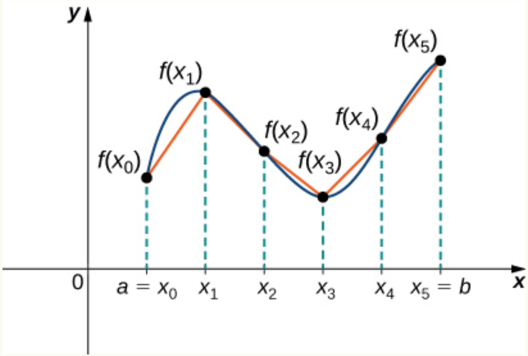
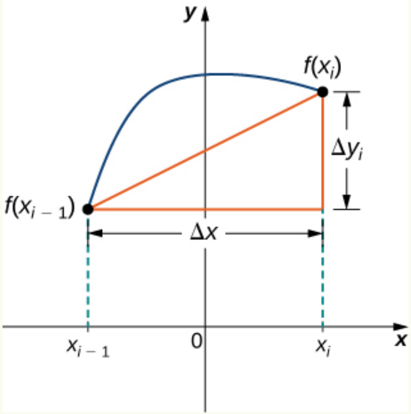
Arc Length

- Let $y = f(x)$ be a **smooth function** defined over $[a, b]$ What is the length of the curve from the point $(a, f(a))$ to the point $(b, f(b))$?





Arc Length (1/2)

- By the **Pythagorean theorem**, the length of the **line segment** is

$$\sqrt{(\Delta x)^2 + (\Delta y_i)^2}.$$

- Factoring out Δx, we obtain

$$\Delta x \sqrt{1 + ((\Delta y_i)/(\Delta x))^2}.$$

- By the **mean value theorem**, there is a point $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = \Delta y_i / \Delta x$.
- Then the length of the line segment is given by

$$\Delta x \sqrt{1 + [f'(x_i^*)]^2}.$$

Arc Length (2/2)

- Adding up the lengths of all the line segments, we get the **arc length**:

$$\text{Arc Length} \approx \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x,$$

which is a **Riemann sum**.

- Taking the limit as $n \to \infty$, we have

$$L[f] := \text{Arc Length} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x = \int_a^b \sqrt{1 + [f'(x)]^2}\,\mathrm{d}x.$$

What is a functional?

- In functional analysis, a **functional** is a mapping from a space of functions into the set of real or complex numbers. That is

$$f \mapsto f(x_0),$$

where x_0 is a parameter.

- The **arc length** is an example of functional, since it can be conceived as a mapping of the function defined by $[a, b]$ to a real number, which is the arc length.

Problem of Variation

Definition 2.1 (Problem of Variation).

The maximization or minimization of a functional $L[f]$, i.e., choosing a function f^* such that $L[f^*]$ is maximum or minimum, is called the **problem of variation**.

- Find a function $y = f(x)$ satisfying the **boundary conditions** $f(x_0) = y_0$ and $f(x_1) = y_1$ that minimizes the **functional**

$$L[f] = \int_{x_0}^{x_1} \ell(x, f, f')\,\mathrm{d}x.$$

- The strategy is to set up a perturbation around $f(x)$ as follows.

$$f_\epsilon(x) = f(x) + \epsilon\eta(x).$$

- Here $\epsilon \ll 1$ and $\eta(x)$ is any arbitrary function.

First-Order Condition

- We also require $f_\epsilon(x)$ to satisfy the **boundary conditions**. Hence $\eta(x_0) = \eta(x_1) = 0$.
- Differentiating $L[f_\epsilon]$ with respect to ϵ, we obtain

$$\begin{aligned}\frac{\mathrm{d}L[f_\epsilon]}{\mathrm{d}\epsilon} &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\int_{x_0}^{x_1} \ell(x, f_\epsilon, f'_\epsilon)\,\mathrm{d}x = \int_{x_0}^{x_1} \frac{\partial}{\partial\epsilon}\ell(x, f_\epsilon, f'_\epsilon)\,\mathrm{d}x \\ &= \int_{x_0}^{x_1} \frac{\partial \ell}{\partial f_\epsilon}\frac{\partial f_\epsilon}{\partial \epsilon} + \frac{\partial \ell}{\partial f'_\epsilon}\frac{\partial f'_\epsilon}{\partial \epsilon}\,\mathrm{d}x \\ &= \int_{x_0}^{x_1} \eta\frac{\partial \ell}{\partial f_\epsilon} + \eta'\frac{\partial \ell}{\partial f'_\epsilon}\,\mathrm{d}x.\end{aligned}$$

- Now, we let $\epsilon = 0$, and if $f(x)$ is that **optimal function**, then the **first-order condition** is

$$\frac{\mathrm{d}L[f]}{\mathrm{d}x} = \int_{x_0}^{x_1} \eta\frac{\partial \ell}{\partial f} + \eta'\frac{\partial \ell}{\partial f'}\,\mathrm{d}x = 0.$$

Integration by Parts

- We perform the **integration by parts** on the second term: $\displaystyle\int_{x_0}^{x_1} \eta' \frac{\partial \ell}{\partial f'}\, \mathrm{d}x$
- Here $u = \dfrac{\partial \ell}{\partial f'}$ and $\mathrm{d}v = \eta'\, \mathrm{d}x$.
- By the **fundamental theorem of calculus**,

$$\int \mathrm{d}v = \int \eta'\, \mathrm{d}x \qquad \Longrightarrow \qquad v = \int \frac{\mathrm{d}\eta}{\mathrm{d}x}\, \mathrm{d}x = \int \mathrm{d}\eta = \eta.$$

- Consequently,

$$\int_{x_0}^{x_1} \eta' \frac{\partial \ell}{\partial f'}\, \mathrm{d}x = \frac{\partial \ell}{\partial f'}\eta\Big|_{x=x_0}^{x_1} - \int_{x_0}^{x_1} v \frac{\mathrm{d}u}{\mathrm{d}x}\, \mathrm{d}x = 0 - \int_{x_0}^{x_1} \eta \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \ell}{\partial f'}\right) \mathrm{d}x.$$

Euler-Lagrange Equation

- We can factor out η to obtain

$$\frac{\mathrm{d}L[f]}{\mathrm{d}x} = \int_{x_0}^{x_1} \eta \left(\frac{\partial \ell}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial \ell}{\partial f'} \right) \right) \mathrm{d}x = 0.$$

- Since the function η is arbitrary, it must be that

$$\frac{\partial \ell}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial \ell}{\partial f'} \right) = 0.$$

- This equation is named after Euler and Lagrange.
- It is a necessary condition for a function f to minimize the functional $L(f)$.

Which Function will Produce the Least Arc Length?

- Noting that $y = f(x)$, the partial derivatives are

$$\frac{\partial L(x,y,y')}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}} \quad \text{and} \quad \frac{\partial L(x,y,y')}{\partial y} = 0.$$

- By substituting these partial derivatives into the **Euler–Lagrange equation**, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{y'(x)}{\sqrt{1+(y'(x))^2}} = 0 \qquad \Longrightarrow \qquad \frac{y'(x)}{\sqrt{1+(y'(x))^2}} = C = \text{constant}$$

$$\Rightarrow y'(x) = \frac{C}{\sqrt{1-C^2}} =: A \qquad \Longrightarrow \qquad y(x) = Ax + B$$

where B is the constant of integration.

- Straight line!

Keywords