Lesson 4
Integration

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Inspiring Quote

Science is the Differential Calculus of the mind. Art the Integral Calculus; they may be beautiful when apart, but are greatest only when combined.

— Ronald Ross

Learning Outcomes

- Develop an intuitive understanding of differentiation versus integration.
- Describe definite integral as area under the curve.
- Apply the geometric sum in integration.
- Explain the fundamental theorem of calculus.
- Apply the integration technique based on the translation theorem.
- Apply the technique of integration by parts.
- Explain the heuristics of LATE.
- Understand how normal distribution can be derived from first principles.

Revision

Definition 1.1 (Antiderivative).

An **antiderivative** $F(x)$ of a function $f(x)$ is a function whose derivative is $f(x)$. In other words, $F'(x) = f(x)$.

Theorem 1.2 (Constant of Integration).

Suppose that $F(x)$ and $G(x)$ are antiderivatives of a function $f(x)$. Then $F(x)$ and $G(x)$ differ only by a constant. That is, $F(x) = G(x) + C$ for some constant C.

So, to find *all* antiderivatives of a function, it is necessary only to find *one* antiderivative and then add a generic constant to it.

Indefinite Integral

Definition 1.3 (Indefinite Integral).

The **indefinite integral** of a function $f(x)$ is denoted by

$$\int f(x)\,\mathrm{d}x.$$

It represents the entire family of **antiderivatives** of $f(x)$.

- The large S-shaped symbol before $f(x)$ is called an **integral sign**.
- $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int f(x)\,\mathrm{d}x\right) = f(x)$.
- For an antiderivative $F(x)$ of a function $f(x)$, the **infinitesimal** $\mathrm{d}F$ is given by

  $\mathrm{d}F = F'(x)\,\mathrm{d}x = f(x)\,\mathrm{d}x$, and so $\boxed{F(x) = \int f(x)\,\mathrm{d}x = \int \mathrm{d}F}$.

Application: Gas Law

- Pressure P, volume V, and temperature T of a tank of gas:

$$\frac{\mathrm{d}P}{P} + \frac{\mathrm{d}V}{V} = \frac{\mathrm{d}T}{T}$$

- Integrating both sides of the equation yields

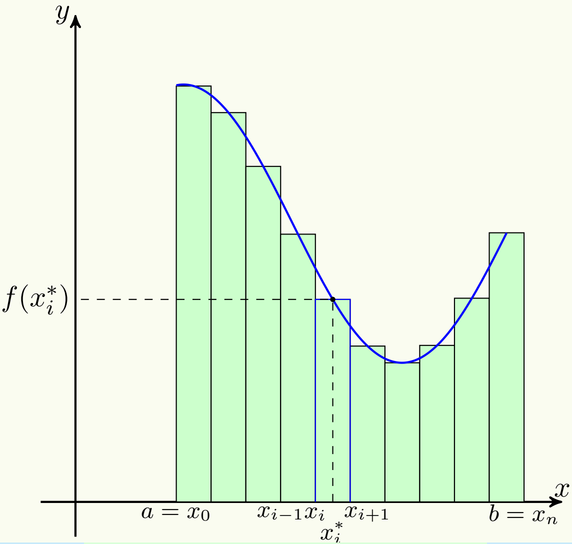
$$\int \frac{\mathrm{d}P}{P} + \int \frac{\mathrm{d}V}{V} = \int \frac{\mathrm{d}T}{T}$$

$$\begin{aligned} \ln P + \ln V &= \ln T + C \quad (C \text{ is a constant}) \\ \ln(PV) &= \ln T + C \\ PV &= e^{\ln T + C} = e^{\ln T} \cdot e^{C} = T\, e^{C} = RT, \end{aligned}$$

where $R = e^C$ is a constant.

- So, $\boxed{PV = RT}$.

Riemann Sum of Areas



What is Definite Integral?

Definition 2.1 (Definite Integral).

The **definite integral** of a continuous function f on the interval $[a, b]$, denoted $\int_a^b f(x)\ \mathrm{d}x$, is the real number given by

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

where $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$ for $i = 0, \ldots, n$.

The points x_i^* satisfy $x_{i-1} \leqq x_i^* \leqq x_i$ for $i = 0, \ldots, n$.

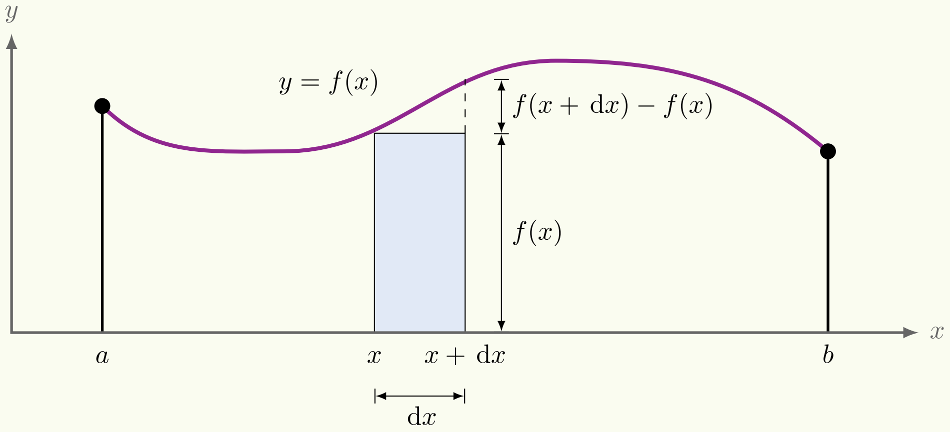
It represents the sum of the **infinitesimals** $f(x)\,\mathrm{d}x$ for all $x \in [a, b]$.

Names

$$\int_a^b f(x)\,\mathrm{d}x$$

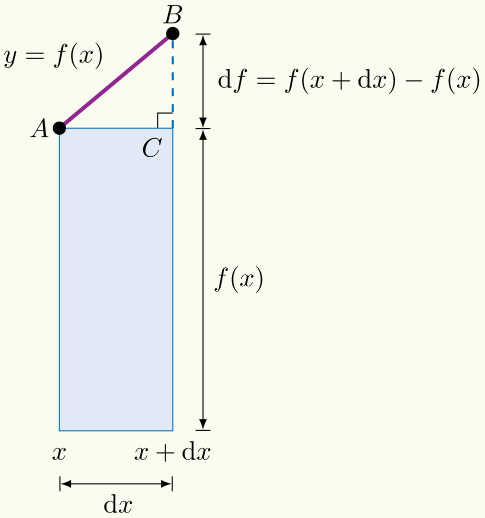
- The numbers a and b are called the **limits of integration**.
- a is the **lower limit of integration**.
- b is the **upper limit of integration**.
- The function $f(x)$ being integrated is called the **integrand**.

Infinitesimal Area



The infinitesimal $f(x)\,\mathrm{d}x$ is the area of a rectangle.

Area under the Curve $y = f(x)$ over $[x, x + \mathrm{d}x]$



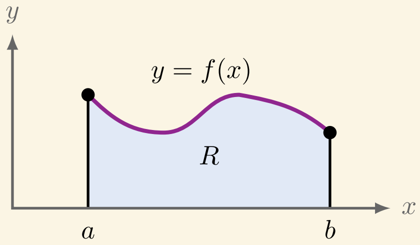
$$\begin{aligned}
\text{Area of } \triangle ABC &= \frac{1}{2}(\text{base}) \times (\text{height}) \\
&= \frac{1}{2}(\mathrm{d}x)(\mathrm{d}f) \\
&= \frac{1}{2}(\mathrm{d}x)(f'(x)\,\mathrm{d}x) \\
&= \frac{1}{2}f'(x)(\mathrm{d}x)^2 \\
&= 0
\end{aligned}$$

Area under the Curve

Definition 2.2 (Area under the Curve).

For a function $f(x) \geqq 0$ over $[a, b]$, the **area under the curve** $y = f(x)$ between $x = a$ and $x = b$, denoted by A, is given by

$$A = \int_a^b f(x)\,\mathrm{d}x$$



and represents the area of the region R bounded above by $y = f(x)$, bounded below by the x-axis, and bounded on the sides by $x = a$ and $x = b$ (with $a < b$).

Example

$$\int_1^2 x^2\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i-1})\,\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^{n} x_{i-1}^2\,\frac{1}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i-1}{n}\right)^2\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{n} + \frac{2}{n^2}(i-1) + \frac{1}{n^3}(i-1)^2\right)$$

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{n} + \frac{2}{n^2}\sum_{i=1}^{n}(i-1) + \frac{1}{n^3}\sum_{i=1}^{n}(i-1)^2\right)$$

$$= \lim_{n\to\infty}\left(1 + \frac{2}{n^2}\sum_{i=1}^{n-1} i + \frac{1}{n^3}\sum_{i=1}^{n-1} i^2\right)$$

$$= \lim_{n\to\infty}\left(1 + \frac{2}{n^2}\cdot\frac{(n-1)n}{2} + \frac{1}{n^3}\cdot\frac{(n-1)n(2n-1)}{6}\right)$$

$$= \left(\lim_{n\to\infty} 1\right) + \left(\lim_{n\to\infty}\frac{n-1}{n}\right) + \left(\lim_{n\to\infty}\frac{2n^2-3n+1}{6n^2}\right) = 1 + \frac{1}{1} + \frac{2}{6} = \boxed{\frac{7}{3}}.$$

Area

Definition 2.3 (Area).

Let R be the region bounded by $y = f(x)$ and the x-axis between $x = a$ and $x = b$. If $f(x) \leqq 0$ over $[a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x \;=\; \text{the negative of the area of } R$$

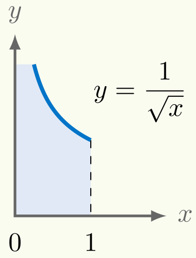
If $f(x)$ changes sign over $[a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x \;=\; \text{the net area of } R,$$

where the parts of R above the x-axis count as positive area and the parts below count as negative area.

A Simple Example

Evaluate $\int_0^1 \frac{dx}{\sqrt{x}}$.



Solution: Since $x = 0$ is a vertical asymptote for $y = \frac{1}{\sqrt{x}}$,

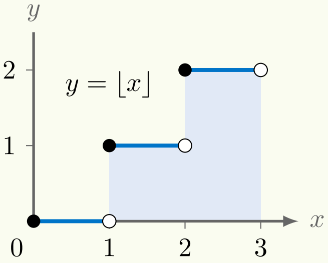
$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{c \to 0+} \int_c^1 \frac{dx}{\sqrt{x}} = \lim_{c \to 0+} \left(2\sqrt{x} \Big|_c^1 \right)$$

$$= \lim_{c \to 0+} (2 - 2\sqrt{c}) = 2 - 0 = 2 \, .$$

This means that the area under the curve $y = 1/\sqrt{x}$ over the interval $(0, 1]$ equals 2. The region is infinite in the y direction.

Step Function

Evaluate $\displaystyle\int_1^3 \lfloor x \rfloor \, dx$.



Solution: The floor function $y = \lfloor x \rfloor$ has jump discontinuities at each integer value of x. The integral needs to be split at the point of discontinuity $x = 2$ within that interval:

$$\begin{aligned}
\int_1^3 \lfloor x \rfloor \, dx &= \int_1^2 \lfloor x \rfloor \, dx + \int_2^3 \lfloor x \rfloor \, dx \\
&= \lim_{b \to 2-} \int_1^b \lfloor x \rfloor \, dx + \lim_{c \to 3-} \int_2^c \lfloor x \rfloor \, dx \\
&= \lim_{b \to 2-} \int_1^b 1 \, dx + \lim_{c \to 3-} \int_2^c 2 \, dx \\
&= \lim_{b \to 2-} \left(x \Big|_1^b \right) + \lim_{c \to 3-} \left(2x \Big|_2^c \right) \\
&= \lim_{b \to 2-} (b - 1) + \lim_{c \to 3-} (2c - 4) = (2 - 1) + (6 - 4) = \boxed{3} .
\end{aligned}$$

The Fundamental Theorem of Calculus

Theorem 3.1 (Fundamental Theorem of Calculus).

Suppose that a function f is differentiable on $[a, b]$.

(I) *The function $A(x)$ defined on $[a, b]$ by*

$$A(x) = \int_a^x f(t)\,\mathrm{d}t$$

is **differentiable** *on $[a, b]$, and*

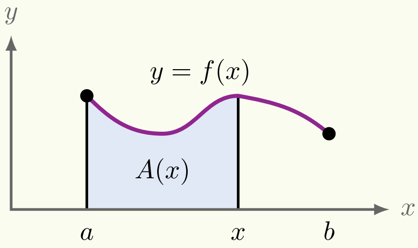
$$A'(x) = f(x)$$

for all x in $[a, b]$.

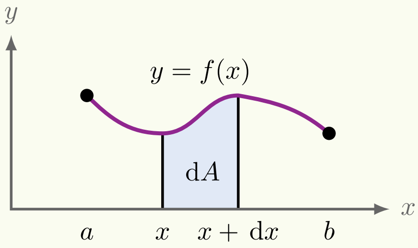
(II) *If F is an* **antiderivative** *of f on $[a, b]$, i.e. $F'(x) = f(x)$ for all x in $[a, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) .$$

Proof of (I)

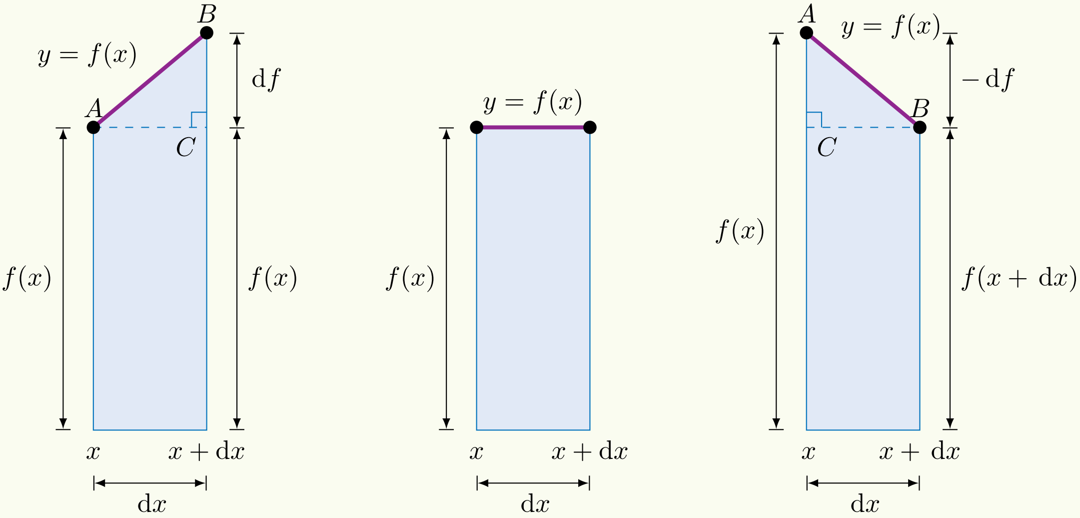


area function $A(x) = \int_a^x f(t)\,\mathrm{d}t$



$$\mathrm{d}A = A(x + \mathrm{d}x) - A(x)$$

Proof of (I) $\mathrm{d}A = f(x)\,\mathrm{d}x$



Proof of (I)

- In the case where f is increasing over $[x, x + \mathrm{d}x]$, the infinitesimal area $\mathrm{d}A$ is the sum of the area of the rectangle of height $f(x)$ and width $\mathrm{d}x$ and the area of the right triangle $\triangle ABC$. The area of $\triangle ABC$ is $\frac{1}{2}(\mathrm{d}f)(\mathrm{d}x) = \frac{1}{2}f'(x)(\mathrm{d}x)^2 = 0$, so $\mathrm{d}A = f(x)\,\mathrm{d}x$.

- In the case where f is constant over $[x, x + \mathrm{d}x]$, the infinitesimal area $\mathrm{d}A$ is the area of the rectangle of height $f(x)$ and width $\mathrm{d}x$. So again, $\mathrm{d}A = f(x)\,\mathrm{d}x$.

- In the case where f is decreasing over $[x, x + \mathrm{d}x]$, the infinitesimal area $\mathrm{d}A$ is the sum of the area of the rectangle of height $f(x + \mathrm{d}x)$ and width $\mathrm{d}x$ and the area of the right triangle $\triangle ABC$. Note that $\mathrm{d}f < 0$ since f is decreasing, and so the area of $\triangle ABC$ is $\frac{1}{2}(-\mathrm{d}f)(\mathrm{d}x) = -\frac{1}{2}f'(x)(\mathrm{d}x)^2 = 0$. Thus,

$$\mathrm{d}A = f(x+\mathrm{d}x)\,\mathrm{d}x = (f(x)+\mathrm{d}f)\,\mathrm{d}x = f(x)\,\mathrm{d}x + f'(x)(\mathrm{d}x)^2 = f(x)\,\mathrm{d}x + 0 = f(x)\,\mathrm{d}x\,.$$

Proof of (I) Conclusion

- So in all three cases,

$$\mathrm{d}A = f(x)\,\mathrm{d}x,$$

- So

$$A'(x) = \frac{\mathrm{d}A}{\mathrm{d}x} = f(x),$$

- It shows that $A(x)$ is differentiable and has derivative $f(x)$.
- This proves Part I of the **Fundamental Theorem of Calculus**.

Proof of (II)

- Let $F(x)$ be an antiderivative of $f(x)$ over $[a, b]$.
- Now $A(x) = \int_a^x f(x)\,\mathrm{d}x$ is also an antiderivative of $f(x)$ over $[a, b]$ by Part I of the theorem.
- So $A(x)$ and $F(x)$ differ by a constant C. In other words,

$$A(x) = F(x) + C \quad \text{for all } x \text{ in } [a, b].$$

- By definition $A(a) = 0$, since it is the area under the curve over the interval $[a, a]$ of zero length. Thus,

$$0 = A(a) = F(a) + C \quad \Rightarrow \quad C = -F(a) \quad \Rightarrow \quad A(x) = F(x) - F(a),$$

for all x in $[a, b]$. So

$$\int_a^b f(x)\,\mathrm{d}x = A(b) = F(b) - F(a).$$

Nice Trick

Theorem 3.2 (Translation).

For any constant a,

$$\int_0^a f(x)\,\mathrm{d}x = \int_0^a f(a-x)\,\mathrm{d}x.$$

Proof.

- Let $u = a - x$, so $x = a - u$, and $\mathrm{d}x = -\mathrm{d}u$.
- Then $x = 0$ becomes $u = a$ and $x = a$ becomes $u = 0$ in the limits of integration:

$$\int_0^a f(x)\,\mathrm{d}x = -\int_a^0 f(a-u)\,\mathrm{d}u = \int_0^a f(a-u)\,\mathrm{d}u = \int_0^a f(a-x)\,\mathrm{d}x$$

□

Example of Definite Integral

Evaluate $\displaystyle\int_0^{\pi} \frac{x\ \sin\ x}{1+\cos^2 x}\,\mathrm{d}x$.

Solution: Let $I = \displaystyle\int_0^{\pi} \frac{x\ \sin\ x}{1+\cos^2 x}\,\mathrm{d}x$. Then by the translation theorem (with $a=\pi$):

$$I = \int_0^{\pi} \frac{(\pi - x)\ \sin(\pi - x)}{1+\cos^2(\pi - x)}\,\mathrm{d}x = \int_0^{\pi} \frac{(\pi - x)\ \sin\ x}{1+\cos^2 x}\,\mathrm{d}x$$

$$= \pi \int_0^{\pi} \frac{\sin\ x}{1+\cos^2 x}\,\mathrm{d}x - \int_0^{\pi} \frac{x\ \sin\ x}{1+\cos^2 x}\,\mathrm{d}x \quad\Longrightarrow\quad I = \pi \int_0^{\pi} \frac{\sin\ x}{1+\cos^2 x}\,\mathrm{d}x - I$$

$$2I = \pi \int_0^{\pi} \frac{\sin\ x}{1+\cos^2 x}\,\mathrm{d}x = -\pi\ \tan^{-1}(\cos\ x)\Big|_0^{\pi} = -\pi\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{\pi^2}{2}$$

$$I = \boxed{\frac{\pi^2}{4}}.$$

Integration by Parts

Theorem 4.1 (Integration by Parts).

For differentiable functions u and v:

$$\int u\,dv = uv - \int v\,du \tag{1}$$

Proof.

- Start with the product rule: $(u\,v)' = u'\,v + u\,v'$.
- Integrate both sides: $\int (u\,v)'dx = \int u'\,v + u\,v'dx$.
- The result: $uv = \int u'\,v\,dx + \int u\,v'\,dx = \int v\,du + \int u\,dv$.

□

Example of LATE

Evaluate the integral $\int_{-1}^{2} x\mathrm{e}^{6x}\,dx$

- Which should be u? Which should be dv?
- LATE choice: **L**ogarithmic **A**lgebraic **T**rigonometric **E**xponential
- So let $u = x$ and $dv = \mathrm{e}^{6x}dx$.
- Hence $du = dx$ and $\int dv = \int \mathrm{e}^{6x}dx \implies v = \frac{1}{6}\mathrm{e}^{6x}$.
- Plug into the formula. We get $\int x\mathrm{e}^{6x}dx = \frac{1}{6}x\mathrm{e}^{6x} - \int \frac{1}{6}\mathrm{e}^{6x}dx = \frac{1}{6}x\mathrm{e}^{6x} - \frac{1}{36}\mathrm{e}^{6x}$.
- So

$$\int_{-1}^{2} x\mathrm{e}^{6x}\,dx = \left(\frac{x}{6}\mathrm{e}^{6x} - \frac{1}{36}\mathrm{e}^{6x}\right)\Bigg|_{-1}^{2} = \left(\frac{1}{3}\mathrm{e}^{12} - \frac{1}{36}\mathrm{e}^{12}\right) - \left(-\frac{1}{6}\mathrm{e}^{-6} - \frac{1}{36}\mathrm{e}^{-6}\right)$$
$$= \frac{11}{36}\mathrm{e}^{12} + \frac{7}{36}\mathrm{e}^{-6}.$$

Normal Probability Density Function

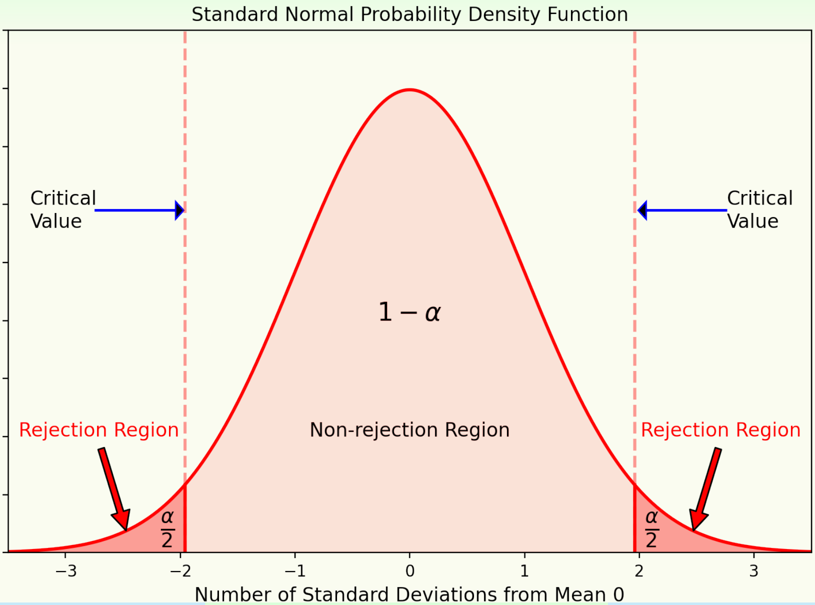
- **Gaussian** or **normal probability density function** $p(x)$ with mean μ and variance σ^2 is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

- With no loss of generality, we can shift the **mean** μ to 0 by a change of variable that corresponds to a simple linear shift operation $x^\sharp = x - \mu$. Then reuse x for the variable of $p(x)$. So

$$\boxed{p(x) = \frac{1}{\sigma\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}.}$$

- Our goal is to derive $p(x)$ from first principles, so as to gain an intuitive understanding of **Gaussian distribution**.


Standard Normal Probability Density Function
Critical Value
Critical Value
1 − α
Non-rejection Region
Rejection Region
Rejection Region
α/2
α/2
−3
−2
−1
0
1
2
3
Number of Standard Deviations from Mean 0

Setup and the Assumption of Independence

- Suppose we release a packet of fine power vertically from a height above the origin of the $x - y$ plane in an infinitely large room of still air.
- Consider the interval Δx between x and $x + \Delta x$.
- The probability for the powder to land in this interval Δx is $p(x)\Delta x$.
- Similarly the probability of power landing in the Δy is $p(y)\Delta y$.
- The joint probability of landing in the infinitesimal area $\Delta x \Delta y$ is, by the assumption of independence,

$$p(x)\Delta x \, p(y)\Delta y.$$

Assumption of Isotropy

- We postulate that this joint probability is equivalent to $q(r)\Delta x\Delta y$, where $q(r)$ is the probability density function that is dependent only on the distance r from the origin $(0, 0)$.
- This is because in the closed room with no ventilation, we may assume that the powder is equally likely to disperse to every direction.
- So in addition to independence, isotropy is also assumed. Consequently,

$$p(x)\Delta x\, p(y)\Delta y = q(r)\Delta x\Delta y.$$

- In other words, under the assumption of isotropy,

$$p(x)p(y) = q(r).$$

Polar Coordinate System

- In the polar coordinate system, $x = r\cos\theta$ and $y = r\sin\theta$, i.e., x and y are functions of r and θ.
- Differentiating both sides with respect to the angle θ, we obtain

$$p(x)\frac{\partial p(y)}{\partial\theta} + p(y)\frac{\partial p(x)}{\partial\theta} = 0. \qquad (2)$$

- By calculus' chain rule, we have

$$\frac{\partial p(y)}{\partial\theta} = \frac{dp(y)}{dy}\,\frac{\partial y(\theta)}{\partial\theta}, \qquad \text{and} \qquad \frac{\partial p(x)}{\partial\theta} = \frac{dp(x)}{dx}\,\frac{\partial x(\theta)}{\partial\theta}.$$

Differential Equation

- Since $\frac{d\sin\theta}{d\theta} = \cos\theta$ and $\frac{d\cos\theta}{d\theta} = -\sin\theta$, we obtain

$$\frac{\partial y(\theta)}{\partial\theta} = r\cos\theta = x \qquad \text{and} \qquad \frac{\partial x(\theta)}{\partial\theta} = -r\sin\theta = -y.$$

- It follows that the differential equation (2) becomes

$$p(x)p'(y)x - p(y)p'(x)y = 0.$$

- Here the prime $'$ refers to differentiation with respect to the function's variable.

Using the Assumption of Independence

- To solve this differentiation equation, we rewrite it as follows:

$$\frac{p'(x)}{xp(x)} = \frac{p'(y)}{yp(y)}.$$

- Since x and y are independent, the ratio defined by the differential equation must necessarily be a constant C. That is,

$$\frac{p'(x)}{xp(x)} = \frac{p'(y)}{yp(y)} = C.$$

- Next, to solve the differential equation, $\dfrac{p'(x)}{xp(x)} = C$, we write

$$\frac{p'(x)}{p(x)} = Cx, \quad \text{equivalently,} \quad \frac{dp}{p} = Cx\,dx.$$

Solution with Two Constants

- The solution is the indefinite integral with the integration constant a, i.e.,

$$\ln\left(p(x)\right) = \frac{C}{2}x^2 + a.$$

- It can be rewritten as, with $A := \mathrm{e}^a$,

$$p(x) = A\exp\left(\frac{C}{2}x^2\right).$$

- From the standpoint of diffusion in dispersing the powder, it is less likely for the density $p(x)$ to be large when x is large, i.e., far away from the origin.

- Therefore, the constant C is necessarily negative. Hence we write $C =: -\zeta^2$, and the probability density function $p(x)$ becomes

$$p(x) = A\mathrm{e}^{-\frac{\zeta^2}{2}x^2}.$$

Square the Integral

- Now, probability must sum to 1.

$$\int_{-\infty}^{\infty} p(x)\, dx = 1.$$

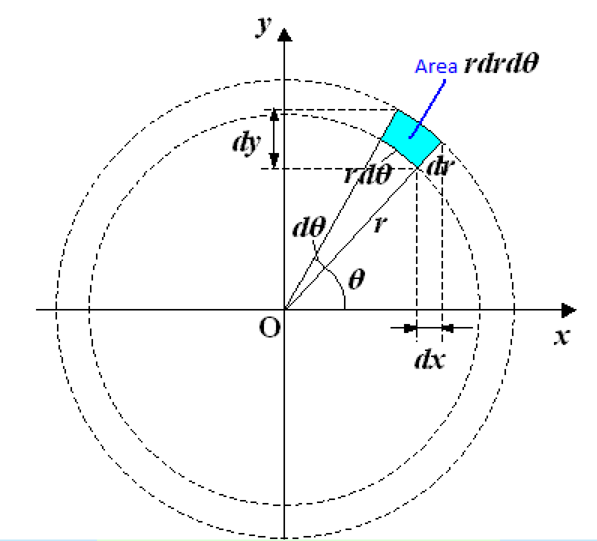
- Since $\mathrm{e}^{-\frac{\zeta^2}{2}x^2}$ is an even function, it follows that

$$\frac{1}{A} = \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\zeta^2}{2}x^2}\, dx = 2\int_{0}^{\infty} \mathrm{e}^{-\frac{\zeta^2}{2}x^2}\, dx.$$

- To change the coordinate system from Cartesian to polar, we square both sides of the equation to obtain

$$\frac{1}{4A^2} = \left(\int_{0}^{\infty} \mathrm{e}^{-\frac{\zeta^2}{2}x^2}\, dx\right) \times \left(\int_{0}^{\infty} \mathrm{e}^{-\frac{\zeta^2}{2}y^2}\, dy\right). \qquad (3)$$

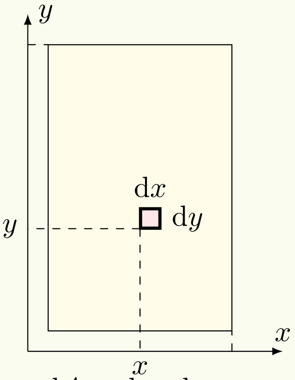
Infinitesimal Area



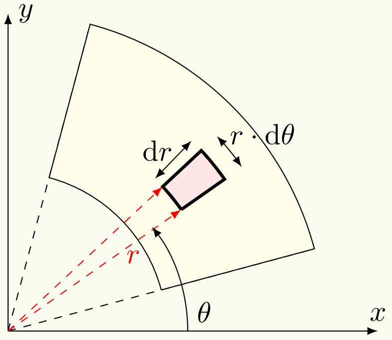
Derivation of Infinitesimal Areas

- The infinitesimal area $\mathrm{A} := \mathrm{d}x \cdot \mathrm{d}y$ is equivalent to $\mathrm{d}r \cdot r\mathrm{d}\theta$.

$$\mathrm{d}A = \frac{1}{2}(r + \mathrm{d}r)^2\mathrm{d}\theta - \frac{1}{2}r^2\mathrm{d}\theta = r\,\mathrm{d}r\,\mathrm{d}\theta + \frac{1}{2}(\mathrm{d}r)^2\,\mathrm{d}\theta = r\,\mathrm{d}r\,\mathrm{d}\theta$$



$$\mathrm{d}A = \mathrm{d}x \cdot \mathrm{d}y$$



$$\mathrm{d}A = r \cdot \mathrm{d}\theta \cdot \mathrm{d}r$$

Change to Polar Coordinates

- Consequently, we obtain, knowing that $r^2 = x^2 + y^2$,

$$\int_0^{\infty}\int_0^{\infty} \mathrm{e}^{-\frac{\zeta^2}{2}\left(x^2+y^2\right)}\, dx\, dy = \int_0^{\frac{\pi}{2}}\int_0^{\infty} \mathrm{e}^{-\frac{\zeta^2}{2}r^2} r\, dr\, d\theta.$$

- The region of integration on the left-hand side is the first quadrant.
- Accordingly, in the polar coordinate system,
  - r ranges from 0 to ∞
  - the angle θ goes from 0° to 90°, which is $\pi/2$.

Integration in Polar Coordinates: A Identified

- Now, we note that the radius r and θ are independent.
- So we can separate the double integral into a product of two single integrals. Hence,

$$
\begin{aligned}
\frac{1}{4A^2} &= \int_0^{\frac{\pi}{2}} \int_0^{\infty} \mathrm{e}^{-\frac{\zeta^2}{2} r^2} r \, dr \, d\theta = \int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} \mathrm{e}^{-\frac{\zeta^2}{2} r^2} r \, dr \\
&= \frac{\pi}{2} \int_0^{\infty} \mathrm{e}^{-\frac{\zeta^2}{2} r^2} d(r^2/2) = \frac{\pi}{2} \int_0^{\infty} \mathrm{e}^{-\zeta^2 z} dz, \qquad \text{where } z := \frac{r^2}{2} \\
&= \frac{\pi}{2} \frac{1}{\zeta^2}.
\end{aligned}
$$

- In this way, we have identified A: $\quad \frac{1}{4A^2} = \frac{\pi}{2\zeta^2} \quad \Longrightarrow \quad \boxed{A = \frac{\zeta}{\sqrt{2\pi}}}.$
- It follows that $\quad p(x) = \frac{\zeta}{\sqrt{2\pi}} \mathrm{e}^{-\frac{\zeta^2}{2} x^2}.$

What is ζ?

- When the mean is zero, the variance is defined as

$$\sigma^2 := \int_{-\infty}^{\infty} x^2 p(x)\, dx = 2\int_0^{\infty} x^2 p(x)\, dx. \tag{4}$$

- With $p(x) = \frac{\zeta}{\sqrt{2\pi}} \mathrm{e}^{-\frac{\zeta^2}{2}x^2}$,

$$\frac{\sigma^2}{2} = \frac{\zeta}{\sqrt{2\pi}} \int_0^{\infty} x^2 \mathrm{e}^{-\frac{\zeta^2}{2}x^2}\, dx.$$

- To performing the integration by parts, we let $u = x$. Hence $du = dx$, and

$$dv = x\,\mathrm{e}^{-\zeta^2\frac{x^2}{2}}\, dx = \mathrm{e}^{-\zeta^2\frac{x^2}{2}}\, d\left(\frac{x^2}{2}\right).$$

$\int u\,dv = uv - \int v\,du$ and ζ Identified

- For the integration $\int dv$, we let $w = \frac{x^2}{2}$, and we obtain $\int \mathrm{e}^{-\zeta^2 w} dw$. It follows that $v = -\frac{1}{\zeta^2}\mathrm{e}^{-\zeta^2 \frac{x^2}{2}}$.

- Therefore, for (4), we have

$$\sigma^2 = 2\int_0^\infty x^2 p(x)dx = \frac{\zeta}{\sqrt{2\pi}}\left(-2x\frac{1}{\zeta^2}\mathrm{e}^{-\zeta^2\frac{x^2}{2}}\Big|_0^\infty + 2\int_0^\infty \frac{1}{\zeta^2}\mathrm{e}^{-\zeta^2\frac{x^2}{2}}dx\right)$$

$$= 0 + \frac{1}{\zeta^2}\left(\frac{\zeta}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\zeta^2\frac{x^2}{2}}dx\right) = \frac{1}{\zeta^2}\times 1$$

$$= \frac{1}{\zeta^2} \quad\Longrightarrow\quad \boxed{\zeta = \frac{1}{\sigma}}.$$

Final Result

- In this way, we have identified ζ to be the reciprocal of standard deviation σ.

- Hence, we derived the normal probability density function (pdf) with mean 0 and variance σ^2:

$$\boxed{p(x) = \frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}.}$$

- When the variance is equal to 1, we obtain the standard normal pdf.

Keywords