Lesson 3
Calculus Theorems and Critical Thinking

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Inspiring Quote

The important thing is not to stop questioning. Curiosity has its own reason for existing.

— Albert Einstein

Learning Outcomes

- Demonstrate the understanding of the concept that the first derivative is the gradient of the function.
- Apply the product rule, quotient rule, and chain rule as if it is your second nature.
- State the derivatives of the inverse trigonometric functions and apply them when finding certain anti-derivatives.
- Establish an intuitive understanding of Rolle's theorem in the context of finding the maximum and minimum of a function.
- Construct a geometric picture of the meaning of mean value theorem.
- Apply L'Hôpital's rule to find the limits of functions.
- Capture the intuitive meanings of Taylor's series and the Maclaurin series.
- Apply the Maclaurin series in examining the validity of radiometric dating with critical thinking.

Gradient

- The **first derivative** of a function $f(x)$ is the limit:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x)}{\Delta},$$

  Here Δ is a very small quantity. If the limit exists, the function $f(x)$ is said to be **differentiable**.

- The first derivative at x is the **gradient** or **slope** of the curve $(x, f(x))$ in a plane.

Derivative of Inverse Trigonometric Functions

- Derivatives of trigonometric functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x) \quad \frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x) \quad \frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \frac{1}{\cos^2(x)}$$

- The inverse trigonometric functions have the following derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \quad \frac{\mathrm{d}}{\mathrm{d}x}\arccos(x) = -\frac{1}{\sqrt{1-x^2}} \quad \frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = \frac{1}{1+x^2}$$

$$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$$

- Let $f(x) = \arcsin(x)$.
- The arc sine is the inverse function of sine. Therefore,

$$\sin(f(x)) = x \qquad \Longrightarrow \qquad \big(\sin(f(x))\big)' = x'.$$

- Apply the chain rule

$$\cos(f(x))f'(x) = 1 \qquad \Longrightarrow \qquad f'(x) = \frac{1}{\cos(f(x))} = \frac{1}{\cos(\arcsin(x))}$$

- From the Pythagorean theorem, we have $\cos(x) = \sqrt{1-\sin^2(x)}$. Therefore the denominator of $f'(x)$ is

$$\cos(\arcsin(x)) = \sqrt{1-\sin^2(\arcsin(x))} = \sqrt{1-x^2}.$$

$$(\arccos(x))' = -\frac{1}{\sqrt{1-x^2}}$$

- Let $f(x) = \arccos(x)$.
- The arc cosine is the inverse function of cosine. Therefore,

$$\cos(f(x)) = x \qquad \Longrightarrow \qquad \big(\cos(f(x))\big)' = x'.$$

- Apply the chain rule

$$-\sin(f(x))f'(x) = 1 \qquad \Longrightarrow \qquad f'(x) = -\frac{1}{\sin(f(x))} = -\frac{1}{\sin(\arccos(x))}$$

- From the Pythagorean theorem, we have $\sin(x) = \sqrt{1-\cos^2(x)}$. Therefore the denominator of $f'(x)$ is

$$\sin(\arccos(x)) = \sqrt{1-\cos^2(\arccos(x))} = \sqrt{1-x^2}.$$

$$(\arctan(x))' = \frac{1}{x^2+1}$$

- Let $f(x) = \arctan(x)$.
- The arc tangent is the inverse function of tangent. Therefore,

$$\tan(f(x)) = x \qquad \Longrightarrow \qquad \big(\tan(f(x))\big)' = x'.$$

- Apply the chain rule

$$\sec^2(f(x))f'(x) = 1 \qquad \Longrightarrow \qquad f'(x) = \frac{1}{\sec^2(f(x))} = \frac{1}{\sec^2(\arctan(x))}$$

- We can write $\sec^2(x) = \tan^2(x) + 1$. Therefore the denominator of $f'(x)$ is

$$\sec^2(\arctan(x)) = \tan^2(\arctan(x)) + 1 = x^2 + 1.$$

Rolle's Theorem

- **Rolle's theorem** states that a **differentiable function** which attains equal values at two distinct points must have a point somewhere between them where the first derivative is zero.
- If a **real-valued function** f is **continuous** on a closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then there exists a c in the open interval (a, b) such that
$$f'(c) = 0.$$
- This version of Rolle's theorem will be used later to prove the mean value theorem.

Strategy behind the Proof of Rolle's theorem

- The idea of the proof is to argue that if $f(a) = f(b)$, then f must attain either a **maximum** or a **minimum** somewhere between a and b, say at c.
- The function must change from increasing to decreasing (or the other way around) at c.
- In particular, if the derivative exists, it must be zero at c.

Proof of Rolle's Theorem

- Suppose that the **maximum** is obtained at an **interior point** c of (a, b).
- We shall examine the above right- and left-hand limits separately.
- For a real h such that $c + h$ is in $[a, b]$, the value $f(c + h)$ is smaller or equal to $f(c)$ because f attains its maximum at c.
- Therefore, for every $h > 0$,

$$\frac{f(c+h) - f(c)}{h} \leqq 0.$$

- Hence

$$f'(c^+) := \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leqq 0,$$

where the limit exists by assumption.

Proof of Rolle's Theorem (Cont'd)

- Similarly, for every $h < 0$, the inequality turns around because the denominator is now negative and we get

$$\frac{f(c+h) - f(c)}{h} \geqq 0.$$

- Hence

$$f'(c^-) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geqq 0.$$

- Finally, when the above right- and left-hand limits agree then the derivative of f at c must be zero.

- The logic for the minimum is similar and the proof is complete.

□

Mean Value Theorem

- The **mean value theorem** states that if a function $f(x)$ is **continuous** on the closed interval $[a, b]$, and **differentiable** on the open interval (a, b), then there exists a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} .$$

- The expression $(f(b) - f(a))/(b - a)$ gives the **slope** of the line joining the points $(a, f(a))$ and $(b, f(b))$, which is a **chord** of the graph of f, while $f'(x)$ gives the slope of the tangent to the curve at the point $(x, f(x))$.

- Thus the mean value theorem implies that given any chord of a smooth curve, we can find a point lying between the end-points of the curve such that the **tangent** at that point is parallel to the chord.

Proof of Mean Value Theorem

- Define $g(x) = f(x) - rx$, where r is a constant. Since f is continuous on $[a, b]$ and differentiable on (a, b), the same is true for g.
- Choose r so that g satisfies the conditions:

$$\begin{aligned} g(a) = g(b) &\iff f(a) - ra = f(b) - rb \\ &\iff r(b-a) = f(b) - f(a) \\ &\iff r = \frac{f(b) - f(a)}{b - a}. \end{aligned}$$

- By Rolle's theorem, since g is continuous and $g(a) = g(b)$, there is some c in (a, b) for which $g'(c) = 0$, and it follows from the equality $g(x) = f(x) - rx$ that

$$f'(c) = g'(c) + r = 0 + r = \frac{f(b) - f(a)}{b - a}.$$

□

What is L'Hôpital's rule?

- **L'Hôpital's rule** is an interesting result to deal with the ratios $0/0$ and $\pm\infty/\infty$ when a certain limit is reached.
- L'Hôpital's rule states that for two functions $f(x)$ and $g(x)$, where $x \in \mathbb{R}$, and a constant value $c \in \mathbb{R}$, if either

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0\,,$$

or

$$\lim_{x \to c} |f(x)| = \lim_{x \to c} |g(x)| = \infty\,,$$

then there exists a limit L to the ratio when x approaches c:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} = L.$$

Proof of L'Hôpital's Rule

- Assume that $g'(x) \neq 0$ is continuous.
- For each x in the interval, define

$$m(x) := \inf\left(\frac{f'(\xi)}{g'(\xi)}\right) \quad \text{and} \quad M(x) := \sup\left(\frac{f'(\xi)}{g'(\xi)}\right),$$

  as ξ ranges over all values between x and c.
- From the differentiability of f and g, the **mean value theorem** ensures that for any two distinct points x and y, there exists a ξ between x and y ($x \neq y$) such that

$$\frac{\dfrac{f(x)-f(y)}{x-y}}{\dfrac{g(x)-g(y)}{x-y}} = \frac{f'(\xi)}{g'(\xi)} \qquad \Longrightarrow \qquad \frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(\xi)}{g'(\xi)}.$$

- Consequently $\quad m(x) \leq \dfrac{f(x)-f(y)}{g(x)-g(y)} \leq M(x)$ for distinct x and y.

Proof of L'Hôpital's Rule Case 1: $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$.

- For any x and point y between x and c,

$$m(x) \leqq \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{\dfrac{f(x)}{g(x)} - \dfrac{f(y)}{g(x)}}{1 - \dfrac{g(y)}{g(x)}} \leqq M(x)\,,$$

- As y approaches c, $\dfrac{f(y)}{g(x)}$ and $\dfrac{g(y)}{g(x)}$ become zero, and so

$$m(x) \leqq \frac{f(x)}{g(x)} \leqq M(x).$$

- Applying the squeeze theorem,

$$\lim_{x \to c} m(x) \leqq \lim_{x \to c} \frac{f(x)}{g(x)} \leqq \lim_{x \to c} M(x) \quad \Longrightarrow \quad \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} = L.$$

Proof of L'Hôpital's Rule Case 2: $\lim_{x\to c} |f(x)| = \lim_{x\to c} |g(x)| = \infty$

- For any x, define $S_x = \{y \mid y \text{ is between } x \text{ and } c\}$.
- For any point y between x and c, we have

$$m(x) \leq \frac{f(y) - f(x)}{g(y) - g(x)} = \frac{\dfrac{f(y)}{g(y)} - \dfrac{f(x)}{g(y)}}{1 - \dfrac{g(x)}{g(y)}} \leq M(x)\,.$$

- As y approaches c, both $\dfrac{f(x)}{g(y)}$ and $\dfrac{g(x)}{g(y)}$ become zero, hence

$$m(x) \leq \liminf_{y\in S_x} \left(\frac{f(y)}{g(y)}\right) \leq \limsup_{y\in S_x} \left(\frac{f(y)}{g(y)}\right) \leq M(x)\,.$$

- The limit supremum and limit infimum are necessary since the existence of the limit of $f(y)/g(y)$ has not yet been established.

Proof of L'Hôpital's Rule Case 2 (Cont'd)

- We need the facts:

$$\lim_{x\to c} m(x) = \lim_{x\to c} M(x) = \lim_{x\to c} \frac{f'(x)}{g'(x)} = L\,,$$

$$\lim_{x\to c}\left(\liminf_{y\in S_x} \frac{f(y)}{g(y)}\right) = \liminf_{x\to c} \frac{f(x)}{g(x)}$$

and

$$\lim_{x\to c}\left(\limsup_{y\in S_x} \frac{f(y)}{g(y)}\right) = \limsup_{x\to c} \frac{f(x)}{g(x)}\,.$$

- The squeeze theorem again asserts that $\liminf_{x\to c} \frac{f(x)}{g(x)} = \limsup_{x\to c} \frac{f(x)}{g(x)} = L$, and so $\lim_{x\to c} \frac{f(x)}{g(x)} = L = \lim_{x\to c} \frac{f'(x)}{g'(x)}$.

□

Example of L'Hôpital's Rule

Find the limit of

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

Solution:

- Apply the l'Hôpital's rule by first differentiating the numerator and denominator with respect to x to obtain

$$\frac{\cos x}{1}.$$

- Therefore as $x \to 0$, the limit is 1:

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \cos x = \boxed{1}.$$

Taylor's Series

- A **Taylor series** is a series expansion of a real function $f(x)$ about a point $x = a$.

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \, .$$

- Here, $f^{(n)}(a)$ is a symbol for

$$\left.\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x)\right|_{x=a} ,$$

which is the value of the n-th derivative of $f(x)$ computed at the point a.

Remarks

- The derivative of order zero is defined to be $f(x)$ itself and $(x-a)^0$ and $0!$ are both defined to be 1.
- In the special case that $a = 0$, the series is called the **Maclaurin series**.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n\,.$$

- Example: $f(x) = \ln(x+1)$

$$\begin{aligned} \ln(x+1) &= 0 + \frac{1}{0+1}x + \frac{1}{2!}\frac{-1}{(0+1)^2}x^2 + \cdots \\ &= x - \frac{1}{2}x^2 + \cdots \end{aligned}$$

Taylor's Theorem

Theorem 5.1 (Taylor's Theorem).

Let $k \geq 1$ be an integer and let the function $f : \mathbb{R} \to \mathbb{R}$ be k times differentiable at the point $a \in \mathbb{R}$. Then there exists a function $h_k : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + h_k(x)(x-a)^k,$$

and

$$\lim_{x \to a} h_k(x) = 0\,.$$

Proof of Taylor's Theorem

- Let $P(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots + \dfrac{f^{(k)}(a)}{k!}(x-a)^k$ and

$$h_k(x) = \begin{cases} \dfrac{f(x) - P(x)}{(x-a)^k} & x \neq a \\ 0 & x = a \end{cases}$$

- Note that $P(a) = f(a)$, $P'(a) = f'(a), \ldots, P^{(k-1)}(a) = f^{(k-1)}(a)$. That is, for each $j = 0, 1, \ldots, k-1$, $f^{(j)}(a) = P^{(j)}(a)$.
- Each of the first $k-1$ derivatives of the numerator in $h_k(x)$ vanishes at $x = a$, and the same is true of the denominator.
- It is sufficient to show that $\lim_{x \to a} h_k(x) = 0$.
- The proof is based on repeated application of L'Hôpital's rule.

Proof of Taylor's Theorem (Cont'd)

$$\lim_{x\to a}\frac{f(x)-P(x)}{(x-a)^k}=\lim_{x\to a}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)-P(x)\big)}{\frac{\mathrm{d}}{\mathrm{d}x}(x-a)^k}=\cdots=\lim_{x\to a}\frac{\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\big(f(x)-P(x)\big)}{\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}(x-a)^k}$$

$$=\frac{1}{k!}\lim_{x\to a}\frac{f^{(k-1)}(x)-P^{(k-1)}(x)}{x-a}$$

$$=\frac{1}{k!}\left(\lim_{x\to a}\frac{f^{(k-1)}(x)-f^{(k-1)}(x-a)}{x-a}-\lim_{x\to a}\frac{P^{(k-1)}(x)-P^{(k-1)}(x-a)}{x-a}\right)$$

$$=\frac{1}{k!}\left(\lim_{x\to a}\frac{f^{(k-1)}(x)-f^{(k-1)}(x-a)}{x-a}-\lim_{x\to a}\frac{f^{(k-1)}(x)-f^{(k-1)}(x-a)}{x-a}\right)$$

$$=\frac{1}{k!}\big(f^{(k)}(a)-f^{(k)}(a)\big)=0\,.$$

The second last equality follows by the definition of the derivative at $x=a$. □

What is Scientific Method?

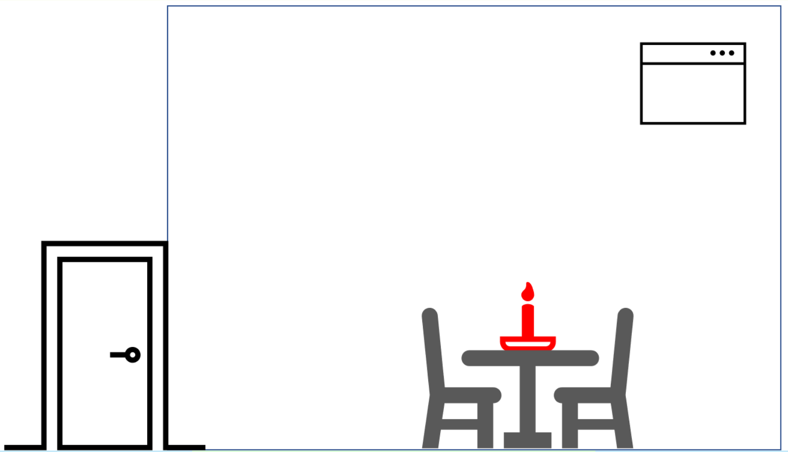
- Believing that there are "**patterns**" to be discovered, we want to know how things work.
- Trusting our instruments and senses, we make observations and gather data.
- Assuming that the "patterns" will not wantonly appear and disappear, we propose a hypothesis/model to explain or describe the patterns.
- Using mathematics, we make predictions for yet to observe "patterns."
- Applying logical induction, we test the predictions through observations over and over again.

What is Science?

- Knowledge of reality generated by objective investigation with scientific method.
- Criteria of science:
  - Observational regularity at present time.
  - Experimental controllability, repeatability, verifiability at present time.
- Pure science: experiments can in principle be performed at any place on earth, moon, outer space, (may be) Mars, at present time.
- Observation science: Not experimental but contingent on the observed regularity of objects/living things under investigation, at present time.
- Is data science a science?
- Is history science?

Thought Experiment

- You enter a room and you see a candle burning.
- For how long has the candle been burning before you enter the room at time t_1?



Common Sense

- You measure the candle's length L_1 at time t_1. Later, you measure the candle's length L_2 again at time t_2, You can then determine the rate α of burning,

$$\alpha = \frac{L_1 - L_2}{t_2 - t_1}.$$

- If you knew the original length L_0 of the candle before it was lit, then you would know for how long has the candle been burning:

$$T = \frac{L_0 - L_1}{\alpha}.$$

- If you don't know L_0, it is not possible to answer the question.

Historical Science versus Non-Historical Science

- What most people don't realize is that there are two broad kinds of scientific enterprises.

- The **historical science** is about past events such as the origins or the beginnings of the universe, the earth, the moon, the sun, the stars, the galaxies, etc. It is also about the origins of first life, consciousness, and mind.

- The **non-historical science** is the usual science that we have learned in school. It is about *current* natural things that are controllable in a laboratory, measurable, experimentally repeatable, and at least in principle observable *now*.

Philosophy Underlying Historical Science

- Uniformitarianism (US **National Park Service**)
  - The past history of our globe must be explained by what can be seen to be happening now.
- Naturalism (**Encyclopedia Britannica**)
  - All beings and events in the universe are natural. Consequently, all knowledge of the universe falls within the pale of scientific investigation.
- Materialism (**Encyclopedia Britannica**)
  - All facts, including facts about the human mind and will and the course of human history, are causally dependent upon physical processes, or even reducible to them.
- Scientism (**The Basics of Philosophy**)
  - The assumptions and methods of research of the physical and natural sciences are equally appropriate—even essential—to all other disciplines. It is based on the belief that natural science has authority over all other interpretations of life, and that the methods of natural science form the only proper elements in any philosophical or other inquiry.

Candle Example

- In the earlier example, the calibration of the rate of burning α is non-historical science.
- If you want to know how long will it take for the whole candle to be burned completely, you can answer this non-historical question by the *forecast* τ:

$$\tau = \frac{L_2 - 0}{\alpha} = \frac{L_2}{\alpha}\,.$$

- But when the question becomes "For how long had the candle been burning?", then it is about the past event when the candle was lit. Then it becomes historical "science."
- On top of being not possible to know the original length L_0, it could happen that the window was open and a strong wind extinguished the candle, and sometime later, somebody lit it again.
- This is the essence of what radiometric dating of a rock is all about.

Radiometric Dating 101

- Element such as uranium 238 $^{238}\mathrm{U}$ doesn't remain unchanged.
- Eventually, every $^{238}\mathrm{U}$ atom will become the lead atom $^{206}\mathrm{Pb}$.

$$^{238}\mathrm{U} \longrightarrow {}^{206}\mathrm{Pb}$$

- In general, a **parent atom** radioactively decays into a **daughter atom**.
- Being totally random, it is impossible to predict which atom in a sample of parent atoms will decay into a daughter atom.
- The rate of decay λ is the average of the amount of parents that have undergone decay over a period of time.

Statistical Law of Radioactive Decay

- Current time t; some time s in the past
- Number P_s of radioactive parent atoms measured at time s
- Number P_t of radioactive parent atoms measured at time t
- The rate at which the decrease of parent atoms due to decay is proportional to the number of parent atoms, i.e. P_t.

$$\frac{\mathrm{d}P_t}{\mathrm{d}t} = -\lambda P_t.$$

- Solve this simple ordinary differential equation by calculus:

$$\int \frac{\mathrm{d}P_t}{P_t} = -\lambda \int \mathrm{d}t \quad \Longrightarrow \quad \ln P_t = -\lambda t + C \quad \Longrightarrow \quad P_t = \mathrm{e}^C \mathrm{e}^{-\lambda t}$$

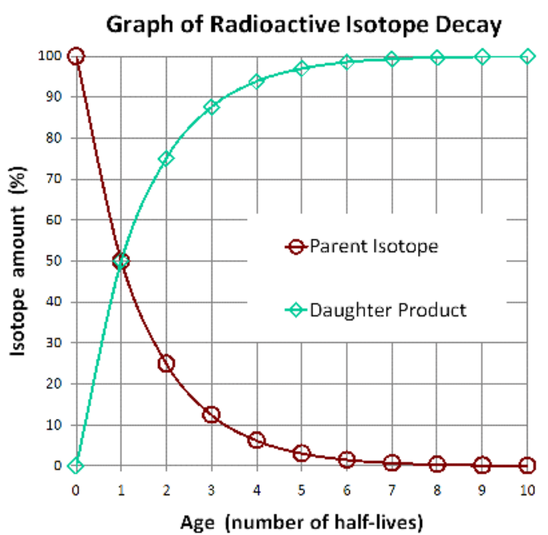
- At time s, $P_s = \mathrm{e}^C \mathrm{e}^{-\lambda s}$. Hence $\mathrm{e}^C = P_s \mathrm{e}^{\lambda s}$, and we obtain

$$P_t = P_s \mathrm{e}^{-\lambda(t-s)}. \tag{1}$$

What is half life?

- How long does it take for the population of parents to be halved by radioactive decay?
- We let $P_t = P_0/2$.
- From (1), $\dfrac{P_0}{2} = P_0\,\mathrm{e}^{-\lambda t} \quad\Longrightarrow\quad \dfrac{1}{2} = \mathrm{e}^{-\lambda t} \quad\Longrightarrow\quad \ln(2) = \lambda t.$
- Therefore, the **half life** is
$$t = \frac{\ln(2)}{\lambda}$$
- If we know the half life t, then
$$\frac{1}{\lambda} = \frac{t}{\ln(2)}. \tag{2}$$

Parents and Daughters



Non-Historical Science

- Result of Calculus:
  Radioactive decay of parent atoms from time s to time t is exponential at the rate of λ:

$$P_t = P_s \, \mathrm{e}^{-\lambda(t-s)}.$$

- You had measured P_s at time s, say 2 years ago.
- You measure P_t today of the same sample kept in the lab.
- Hence you can calibrate λ by counting the *parents only* at two different times:

$$\lambda = -\frac{1}{t-s} \ln\left(\frac{P_t}{P_s}\right).$$

- Note that the **decay rate** $\lambda > 0$, since $P_t < P_s$, $\dfrac{P_t}{P_s} < 1$, hence $\ln\left(\dfrac{P_t}{P_s}\right) < 0$.

Historical Science

- Now that you have calibrated λ.
- You want to know when the rock sample was formed in the *historical past*, i.e., $s = 0$.
- You then use the same exponential decay with an unknown number P_0 of parent atoms:

$$P_t = P_0 \, e^{-\lambda t}. \tag{3}$$

- DesertUSA: **"The number of parent atoms originally present is simply the number present now plus the number of daughter atoms formed by the decay, both of which are quantities that can be measured."**

$$P_0 = P_t + D_t.$$

- You can measure the number of daughter atoms D_t today as well. Hence geologists assume that $P_0 = P_t + D_t$ is valid.

BIG Assumption

- By measuring P_t and D_t today, the "age of the rock" can be determined as

$$t = -\frac{1}{\lambda} \ln\left(\frac{P_t}{P_t + D_t}\right). \tag{4}$$

- Do you see a fatal fallacy in this historical science?
- BIG Assumption: **At time 0, no daughter atom was present. i.e., $D_0 = 0$**.
- But, daughter element, being stable, and atomically simpler than the parent atom, should be present originally.
- Being in the historical past, theoretically and practically, you cannot go back in time to measure the amount of parent and daughter atoms at time 0.

Another BIG Assumption

- The window in the room isn't closed. The wind could blow out the candle; somebody was in the room before you and lit it up again at some unknown time in the past.

- BIG assumption of **radiometric dating**:
  A rock sampled from an open environment subject to the elements of weather, climate changes etc over "x billion years" will not change the initial values of the daughter-to-parent ratios or other ratios into another values.

Yet Another BIG Assumption

- In the candle example, the rate α of burning is assumed to be a constant.
- The wax in the candle may not be homogeneous. Also, the wax near the flame may be partially melted and then re-solidifies below.
- BIG Assumption: **The radioactive decay rate must have been constant in the past at today's measured rate.**
- Nobody can predict when a given parent atom in a sample is going to decay. The rate may not be constant over billions of years.

When Daughter Atoms Were Present Originally

- Now, we rewrite the assumption as

$$P_t = P_0 - D_t.$$

- From (4), we obtain

$$t = -\frac{1}{\lambda}\ln\left(\frac{P_0 - D_t}{P_0}\right) = -\frac{1}{\lambda}\ln\left(1 - \frac{D_t}{P_0}\right).$$

- In general, D_t consists of two parts:
  - The amount D_0 of daughter atoms that exist since the beginning.
  - The amount $D_{t'}$ of daughter atoms produced by the parents when they began to decay.
- Consequently,

$$t = -\frac{1}{\lambda}\ln\left(1 - \frac{D_0 + D_{t'}}{P_0}\right).$$

Fundamental Fallacy of Radiometric Dating

- At the first order of the **Maclaurin series** $\ln(1+x) = x + \cdots$,

$$t \approx \frac{1}{\lambda}\frac{D_0}{P_0} + \frac{1}{\lambda}\frac{D_{t'}}{P_0}. \tag{5}$$

- What some geologists assume is that $D_0 = 0$ and that $t' = t$. They then claim that the age is given by

$$t = t' \approx \frac{1}{\lambda}\frac{D_{t'}}{P_0}$$

- But daughter elements should in general be present originally. Thus, with $D_0 \neq 0$, the true age T should be, from (5),

$$T \approx \frac{1}{\lambda}\frac{D_{t'}}{P_0} \approx t - \frac{1}{\lambda}\frac{D_0}{P_0}. \tag{6}$$

- Since D_0 and P_0 are unknown, theoretically and practically, the age T of a rock cannot be determined by this radiometric method.

A Numerical Illustration

- Consensus measurement of the earth's age: $t = 4.543$ billion years.
- $^{238}\mathrm{U}$ has a half life of 4.468×10^9 years. From (2), $\dfrac{1}{\lambda} = \dfrac{4.468 \times 10^9}{\ln(2)} = 6.446 \times 10^9$.
- But the "true" age is, from (6),

$$T \approx 4.543 \times 10^9 - 6.446 \times 10^9 \frac{D_0}{P_0}.$$

- If $\dfrac{D_0}{P_0} = 0.70478185$, you get $T \approx 6{,}000$ years.
- You can get any age T you want by giving a different value to the unobservable $\dfrac{D_0}{P_0}$.

Marginal Analysis of the Measurement Error in λ

- As in all measurements, infinite precision is impossible. Uncertainty of $|\delta\lambda| \neq 0$ is sure to arise.
- To conduct the **marginal analysis**, we write (4) as

$$t = \frac{1}{\lambda} \ln\left(\frac{P_t + D_t}{P_t}\right).$$

- Then,

$$\frac{\delta t}{\delta\lambda} \approx \frac{\mathrm{d}t}{\mathrm{d}\lambda} = -\frac{1}{\lambda^2} \ln\left(\frac{P_t + D_t}{P_t}\right).$$

- Consequently,

$$|\delta t| \approx \frac{1}{\lambda^2} \ln\left(\frac{P_t + D_t}{P_t}\right) |\delta\lambda| .$$

Numerical Illustration of Marginal Analysis (1)

- We have
$$\frac{1}{\lambda} = \frac{4.468 \times 10^9}{\ln(2)} = 6.446 \times 10^9.$$
- It follows that
$$\frac{1}{\lambda^2} = 4.155 \times 10^{19}.$$
- From Slide 45, we see that Earth's age and the half life of $^{238}\mathrm{U}$ are of the same order of magnitude. That is, for every pair of $^{238}\mathrm{U}$ atoms, one of them has decayed into a daughter atom.
- Based on the BIG assumption of $D_0 = 0$, suppose the ratio measured today $\dfrac{D_t}{P_t} \approx 1$.
- It follows that
$$\ln\left(\frac{P_t + D_t}{P_t}\right) = \ln\left(1 + \frac{D_t}{P_t}\right) \approx \ln(1+1) \approx 0.693.$$

Numerical Illustration of Marginal Analysis (2)

- Hence,

$$\delta t \approx \pm 2.880 \times 10^{19} \times |\delta\lambda|.$$

- Even if you could measure λ at the accuracy of 10^{-9}, the uncertainty δt is

$$\delta t \approx \pm 28.80 \times 10^{9} \quad \text{years.}$$

- The **uncertainty** is twice larger ("older) than the supposed age of the universe, which is believed to be about 14 billion years.
- Does radiometric dating make sense with such uncertainty?
- So, how can you rationally believe what geologists say, "The Earth is 4.5 billion years old."?

Keywords