Session 2
Limits and Derivatives

Christopher Ting

http://cting.x10host.com/

Hiroshima University

✉: cting@hiroshima-u.ac.jp

☎: +81 082-424-6451

A1棟 131-1

June 13, 2025 Version 1.0

Table of Contents

Inspiring Quote

The calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more unequivocally than anything else the inception of modern mathematics; and the system of mathematical analysis, which is its logical development, still constitutes the greatest technical advance in exact thinking.

— John von Neumann

Learning Outcomes

- Recall and review the concepts of limit, continuity, etc.
- Elaborate and grasp the notion of “does not exist”.
- Apply the arithmetic of limits and squeeze theorem.
- Illustrate the intermediate value theorem and how a root of an equation can be found through the method of bisection.
- Recall and revise derivatives and their notations.
- Analyze the concept of chain rule and its proof.
- Memorize and apply the standard forms of derivatives and anti-derivatives.

What is Limit?

Definition 2.1 (Informal Definition of Limit).

We write

$$\lim_{x \to a} f(x) = L$$

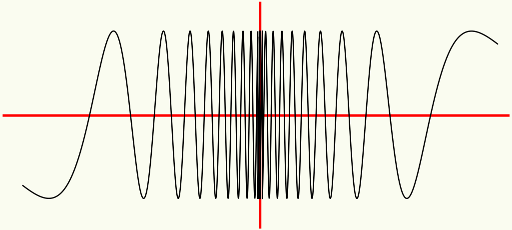
if the value of the function $f(x)$ is sure to be arbitrarily close to L whenever the value of x is close enough to a, without being exactly a.

- Let $f(x) = \dfrac{x-2}{x^2+x-6}$ and consider its limit as $x \to 2$.
- Plug in some numbers close to 2 and see what we find

x	1.9	1.99	1.999	$\cdots$	2.001	2.01	2.1
$f(x)$	0.20408	0.20040	0.20004	$\cdots$	0.19996	0.19960	0.19608

Does Not Exist (DNE)

- Consider the following function $f(x) = \sin(\pi/x)$. Find the limit as $x \to 0$ of $f(x)$.
- We should see something interesting happening close to $x = 0$ because $f(x)$ is undefined there.



- Since the function does not approach a single number as we bring x closer and closer to zero, the limit does not exist.

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = \text{DNE} \quad (\textbf{does not exist})$$

One-Sided Limits

Definition 2.2 (One-Sided Limits).

- We write
$$\lim_{x \to a^-} f(x) = K$$
This is the **left-hand limit** since x approaches a from the left.
- Similarly, we write
$$\lim_{x \to a^+} f(x) = L$$
This is the **right-hand limit** since x approaches a from the right.

Formal Definition of Limit

Definition 2.3.

We say that $\lim_{x\to a} f(x) = L$, if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$. Here L is the **limit point**.

Theorem 2.4.

$\lim_{x\to a} f(x) = L$ *if and only if* $\lim_{x\to a^-} f(x) = L$ *and* $\lim_{x\to a^+} f(x) = L$

Example: Piecewise Function

- Consider a **piecewise function**:

$$f(x) = \begin{cases} x^2 + 5, & \text{if } x \leqq -2; \\ 1 - 3x, & \text{if } x \geqq -2. \end{cases}$$

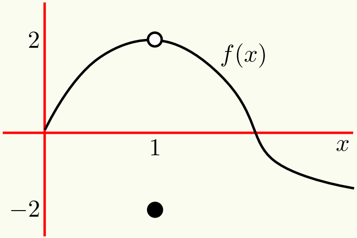
- The left hand and right hand limits are, respectively,

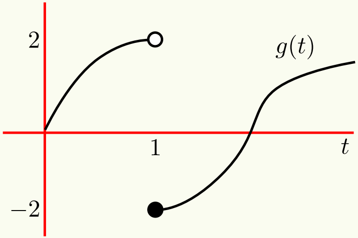
$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} x^2 + 5 = 9\,;$$
$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} 1 - 3x = 7\,.$$

- Therefore, $\lim_{x \to -2} f(x)$ does not exist.

Left and Right Limits

Consider the following two functions and compute their limits and one-sided limits as x approaches 1:





$$\lim_{x\to 1^-} f(x) = 2 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = 2 \quad \Longrightarrow \quad \lim_{x\to 1} f(x) = 2$$

$$\lim_{t\to 1^-} g(t) = 2 \quad \text{and} \quad \lim_{t\to 1^+} g(t) = -2 \quad \Longrightarrow \quad \lim_{t\to 1} g(t) = \text{DNE}$$

Unbounded Limit

Definition 2.5 (Unbounded Limit).

- We write

$$\lim_{x \to a} f(x) = +\infty$$

when the value of the function $f(x)$ becomes arbitrarily large and positive as x gets closer and closer to a, without being exactly a.

- Similarly, we write

$$\lim_{x \to a} f(x) = -\infty$$

when the value of the function $f(x)$ becomes arbitrarily large and negative as x gets closer and closer to a, without being exactly a.

A Note on Unbounded Limit

- Do not think of "$+\infty$" and "$-\infty$" as numbers.
- The statement

$$\lim_{x \to a} f(x) = +\infty$$

does not say "the limit of $f(x)$ as x approaches a is positive infinity." It says "the function $f(x)$ becomes arbitrarily large as x approaches a."

- The statement

$$\lim_{x \to a} f(x) = -\infty$$

says "the function $f(x)$ becomes arbitrarily small as x approaches a."

Arithmetic of Limits

Theorem 2.6 (Arithmetic of Limits).

Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x's that lie in some interval about a (but f, g need not be defined exactly at a). Suppose

$$\lim_{x\to a} f(x) = F \qquad \text{and} \qquad \lim_{x\to a} g(x) = G.$$

exist with $F, G \in \mathbb{R}$. Then the following limits hold:

- $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = F \pm G$
- $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x) = cF$
- $\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = F \cdot G$
- *If* $G \neq 0$ *then* $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{F}{G}$.

Arithmetic of Limits for Powers and Roots

Theorem 2.7 (More Arithmetic of Limits—Powers and Roots).

- *Let n be a positive integer, let $a \in \mathbb{R}$ and let f be a function so that*

$$\lim_{x \to a} f(x) = F$$

for some real number F. Then the following holds

$$\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n = F^n$$

so that the limit of a power is the power of the limit.

- *Similarly, if*
  - n *is an even number and* $F > 0$, *or*
  - n *is an odd number and* F *is any real number*

then $\lim_{x \to a} (f(x))^{1/n} = \left(\lim_{x \to a} f(x)\right)^{1/n} = F^{1/n}$

Squeeze Theorem

Theorem 2.8 (Squeeze or Sandwich Theorem).

Let I be an interval having the point a as a limit point. Let f, g, and h be functions defined on I, except possibly at a itself. Suppose that for every x in I not equal to a, we have

$$g(x) \leq f(x) \leq h(x)\,,$$

and also suppose that

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L\,.$$

Then

$$\lim_{x\to a} f(x) = L\,.$$

Proof of Theorem 2.8

- When $x \to a$, and under the assumption that $g(x) \leq f(x)$, we have

$$\lim_{x \to a} g(x) \leq \liminf_{x \to a} f(x)$$

by the definition of greatest lower bound $\inf$.

- When $x \to a$, and under the assumption that $f(x) \leq h(x)$, we have

$$\limsup_{x \to a} f(x) \leq \lim_{x \to a} h(x)$$

by the definition of least upper bound $\sup$.

- Obviously, we have $\liminf_{x \to a} f(x) \leq \limsup_{x \to a} f(x)$.
- Putting these inequalities together, we obtain

$$L = \lim_{x \to a} g(x) \leq \liminf_{x \to a} f(x) \leq \limsup_{x \to a} f(x) \leq \lim_{x \to a} h(x) = L\,.$$

- Consequently all the inequalities are indeed equalities and the theorem immediately follows. □

Example of Applying Squeeze Theorem

- The limit $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$ cannot be ascertained through the limit law $\lim_{x\to a}\left(f(x)\cdot g(x)\right) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$, because $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist.
- However, by the definition of the sine function,

$$-1 \leqq \sin\left(\frac{1}{x}\right) \leqq 1\,.$$

Multiplying each term by a positive number x^2, it follows that

$$-x^2 \leqq x^2\sin\left(\frac{1}{x}\right) \leqq x^2\,.$$

- Since $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$, by the squeeze theorem, $\lim_{x\to 0} x^2\sin\left(\frac{1}{x}\right)$ must also be 0.

Example 3.1

Example 3.1.

- Compute the following limit: $\lim_{x\to\infty} \dfrac{x^2 - 3x + 4}{3x^2 + 8x + 1}$.
- As x becomes very large, it is the x^2 term that will dominate in both the numerator and denominator, i.e., x^2 is much much larger than x or any constant.

$$\frac{x^2 - 3x + 4}{3x^2 + 8x + 1} = \frac{x^2\left(1 - \dfrac{3}{x} + \dfrac{4}{x^2}\right)}{x^2\left(3 + \dfrac{8}{x} + \dfrac{1}{x^2}\right)}$$

$$= \frac{1 - \dfrac{3}{x} + \dfrac{4}{x^2}}{3 + \dfrac{8}{x} + \dfrac{1}{x^2}}$$ remove the common factors

Example 3.1 (Cont'd)

Example 3.1 (Cont'd)

$$\lim_{x\to\infty}\frac{x^2-3x+4}{3x^2+8x+1}=\lim_{x\to\infty}\frac{1-\dfrac{3}{x}+\dfrac{4}{x^2}}{3+\dfrac{8}{x}+\dfrac{1}{x^2}}$$

$$=\frac{\lim\limits_{x\to\infty}\left(1-\dfrac{3}{x}+\dfrac{4}{x^2}\right)}{\lim\limits_{x\to\infty}\left(3+\dfrac{8}{x}+\dfrac{1}{x^2}\right)}$$ arithmetic of limits

$$=\frac{\lim\limits_{x\to\infty}1-\lim\limits_{x\to\infty}\dfrac{3}{x}+\lim\limits_{x\to\infty}\dfrac{4}{x^2}}{\lim\limits_{x\to\infty}3+\lim\limits_{x\to\infty}\dfrac{8}{x}+\lim\limits_{x\to\infty}\dfrac{1}{x^2}}=\frac{1+0+0}{3+0+0}=\boxed{\frac{1}{3}}.$$

Example 3.2

Example 3.2.

Find the limit as $x \to \infty$ of $\dfrac{\sqrt{4x^2+1}}{5x-1}$

- The denominator is dominated by $5x$.
- The biggest contribution to the numerator comes from $4x^2$.
- When we pull x^2 outside the square-root it becomes x, so the numerator is dominated by $x \cdot \sqrt{4} = 2x$

$$\sqrt{4x^2+1} = \sqrt{x^2(4+1/x^2)} = \sqrt{x^2}\sqrt{4+1/x^2} = x\sqrt{4+1/x^2}.$$

- Thus the limit as $x \to \infty$ is

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{5x-1} = \lim_{x\to\infty} \frac{x\sqrt{4+1/x^2}}{x(5-1/x)} = \lim_{x\to\infty} \frac{\sqrt{4+1/x^2}}{5-1/x} = \boxed{\frac{2}{5}}.$$

Example 3.3

Example 3.3.

Find the limit as $x \to -\infty$ of $\dfrac{\sqrt{4x^2+1}}{5x-1}$

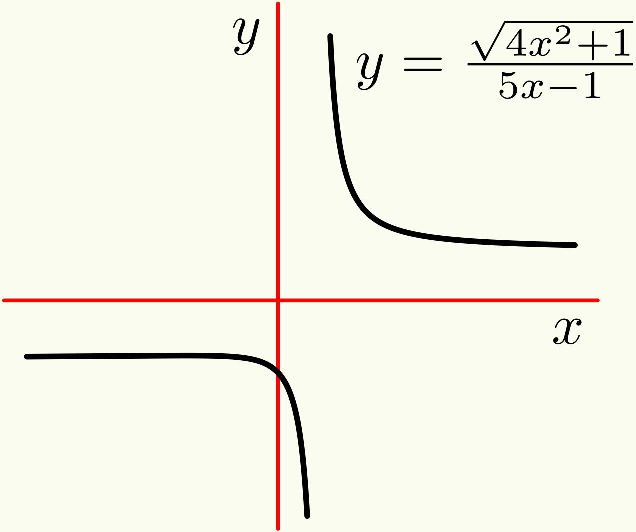
- The biggest contribution to the numerator comes from $4x^2$.
- When we pull x^2 outside a square-root it becomes $|x| = -x$, since we are taking the limit as $x \to -\infty$. Hence,

$$\begin{aligned}\sqrt{4x^2+1} &= \sqrt{x^2(4+1/x^2)} = \sqrt{x^2}\sqrt{4+1/x^2} \\ &= |x|\sqrt{4+1/x^2} = -x\sqrt{4+1/x^2}.\end{aligned}$$

- Thus the limit as $x \to -\infty$ is

$$\lim_{x\to -\infty} \frac{\sqrt{4x^2+1}}{5x-1} = \lim_{x\to -\infty} \frac{-x\sqrt{4+1/x^2}}{x(5-1/x)} = \lim_{x\to -\infty} \frac{-\sqrt{4+1/x^2}}{5-1/x} = \boxed{-\frac{2}{5}}.$$

Graph of Rational Function



Example 3.4

Example 3.4.

- Compute the limit: $\lim_{x\to\infty}\left(x^{7/5}-x\right)$.
- In this case we cannot use the arithmetic of limits to write this as

$$\begin{aligned}\lim_{x\to\infty}\left(x^{7/5}-x\right) &= \left(\lim_{x\to\infty}x^{7/5}\right)-\left(\lim_{x\to\infty}x\right)\\ &= \infty-\infty\end{aligned}$$

  because the limits do not exist.
- We can only use the limit laws when the limits exist.
- When x is very large, $x^{7/5}=x\cdot x^{2/5}$ will dominate the x term.
- So factor out $x^{7/5}$ and rewrite it as $x^{7/5}-x=x^{7/5}\left(1-\frac{1}{x^{2/5}}\right)$.

Example 3.4 (Cont'd)

Example 3.4 (Cont'd)

Consider what happens to each of the factors as $x \to \infty$.

- For large x, $x^{7/5} > x$ (this is actually true for any $x > 1$). In the limit as $x \to +\infty$, x becomes arbitrarily large and positive, and $x^{7/5}$ must be bigger still, so it follows that $\lim_{x\to\infty} x^{7/5} = +\infty$.
- On the other hand, $(1 - x^{-2/5})$ becomes closer and closer to 1. We can use the arithmetic of limits to write this as

$$\lim_{x\to\infty}(1 - x^{-2/5}) = \lim_{x\to\infty} 1 - \lim_{x\to\infty} x^{-2/5} = 1 - 0 = 1.$$

- So the product of these two factors will be come larger and larger (and positive) as x moves off to infinity. Hence we have $\lim_{x\to\infty} x^{7/5}\left(1 - 1/x^{2/5}\right) = +\infty$.

Arithmetic of Infinite Limits

Theorem 3.5 (Arithmetic of Infinite Limits).

Let $a, c, H \in \mathbb{R}$ and let f, g, h be functions defined in an interval around a (but they need not be defined at $x = a$), so that

$$\lim_{x\to a} f(x) = +\infty \qquad \lim_{x\to a} g(x) = +\infty \qquad \lim_{x\to a} h(x) = H$$

- $\displaystyle\lim_{x\to a}(f(x) + g(x)) = +\infty$
- $\displaystyle\lim_{x\to a}(f(x) - g(x))$ *undetermined*
- $\displaystyle\lim_{x\to a}(f(x) + h(x)) = +\infty$
- $\displaystyle\lim_{x\to a}(f(x) - h(x)) = +\infty$
- $\displaystyle\lim_{x\to a} cf(x) = \begin{cases} +\infty & c > 0 \\ 0 & c = 0 \\ -\infty & c < 0 \end{cases}$

Arithmetic of Infinite Limits (Cont'd)

Theorem 3.5 (Cont'd)

- $\lim_{x\to a}(f(x)\cdot g(x)) = +\infty$.
- $\lim_{x\to a} f(x)h(x) = \begin{cases} +\infty & H>0 \\ -\infty & H<0 \\ \text{undetermined} & H=0 \end{cases}$
- $\lim_{x\to a} \frac{f(x)}{g(x)}$ undetermined
- $\lim_{x\to a} \frac{f(x)}{h(x)} = \begin{cases} +\infty & H>0 \\ -\infty & H<0 \\ \text{undetermined} & H=0 \end{cases}$
- $\lim_{x\to a} \frac{h(x)}{f(x)} = 0$

What is continuity?

Definition 4.1 (continuity).

- A function $f(x)$ is **continuous** at a if

$$\lim_{x \to a} f(x) = f(a)$$

- If a function is not continuous at a then it is said to be **discontinuous** at a.
- When we write that f is continuous without specifying a point, then typically we mean that f is continuous at a for all $a \in \mathbb{R}$.
- When we write that $f(x)$ is continuous on the open interval (a, b) then the function is continuous at every point c satisfying $a < c < b$.

Continuity from the Left and from the Right

Definition 4.2 (Continuity from the left and from the right).

- A function $f(x)$ is continuous from the right at a if

$$\lim_{x\to a^+} f(x) = f(a).$$

- Similarly a function $f(x)$ is continuous from the left at a if

$$\lim_{x\to a^-} f(x) = f(a).$$

- A function $f(x)$ is continuous on the closed interval $[a, b]$ when
  - $f(x)$ is continuous on (a, b),
  - $f(x)$ is continuous from the right at a and $f(x)$ is continuous from the left at b.

$$\lim_{x\to a^+} f(x) = f(a) \qquad \text{and} \qquad \lim_{x\to b^-} f(x) = f(b).$$

Arithmetic of Continuity

Theorem 4.3 (Arithmetic of Continuity).

Let $a, c \in \mathbb{R}$ and let $f(x)$ and $g(x)$ be functions that are continuous at a. Then the following functions are also continuous at $x = a$:

- $f(x) + g(x)$
- $f(x) - g(x)$
- $cf(x)$
- $f(x)g(x)$
- $\dfrac{f(x)}{g(x)}$ *provided* $g(a) \neq 0$.

Arithmetic of Continuity

Theorem 4.4 (Composition and Continuity).

If f is continuous at b and $\lim_{x\to a} g(x) = b$ then $\lim_{x\to a} f(g(x)) = f(b)$. That is,

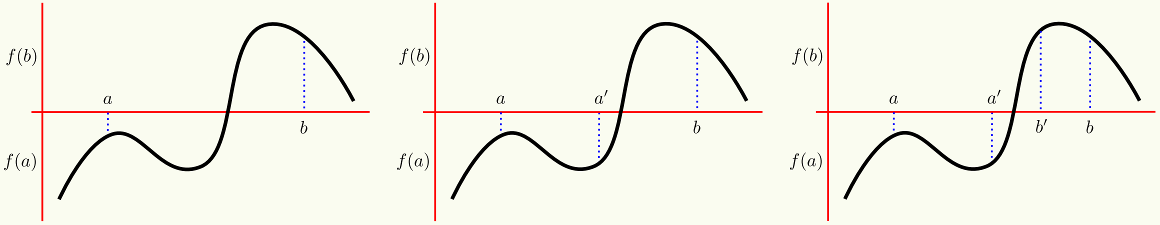
$$\lim_{x\to a} f\left(g(x)\right) = f\left(\lim_{x\to a} g(x)\right).$$

Hence if g is continuous at a and f is continuous at $g(a)$ then the composite function $(f \circ g)(x) = f(g(x))$ is continuous at a.

Theorem 4.5 (Intermediate Value Theorem (IVT)).

Let $a < b$ and let f be a function that is continuous at all points $a \leq x \leq b$. If Y is any number between $f(a)$ and $f(b)$ then there exists some number $c \in [a, b]$ so that $f(c) = Y$.

Application of IVT: Finding a Root



Example 4.6 (Illustration of Bisection Method).

Use the bisection method to find a zero of

$$f(x) = x - 1 + \sin(\pi x/2)$$

that lies between 0 and 1.

Example 4.6 Bisection Method

- We start with the two points $a = 0, b = 1$ and we find that

$$f(0) = -1 \qquad f(1) = 1.$$

- Test the point in the middle $x = \dfrac{0+1}{2} = 0.5$

$$f(0.5) = 0.2071067813 > 0.$$

- So our new interval will be $[0, 0.5]$ since the function is negative at $x = 0$ and positive at $x = 0.5$.

Example 4.6 Bisection Method (Cont'd)

- We now begin with points $a = 0, b = 0.5$ where $f(0) < 0$ and $f(0.5) > 0$.
- Test the point in the middle $x = \dfrac{0 + 0.5}{2} = 0.25$.

$$f(0.25) = -0.3673165675 < 0.$$

- So our new interval will be $[0.25, 0.5]$ since the function is negative at $x = 0.25$ and positive at $x = 0.5$.

Example 4.6 Bisection Method (Cont'd)

- We now begin with points $a = 0.25, b = 0.5$ where $f(0.25) < 0$ and $f(0.5) > 0$.
- Test the point in the middle $x = \dfrac{0.25 + 0.5}{2} = 0.375$.

$$f(0.375) = -0.0694297669 < 0.$$

- So our new interval will be $[0.375, 0.5]$ since the function is negative at $x = 0.375$ and positive at $x = 0.5$.

Example 4.6 Bisection Method (Cont'd)

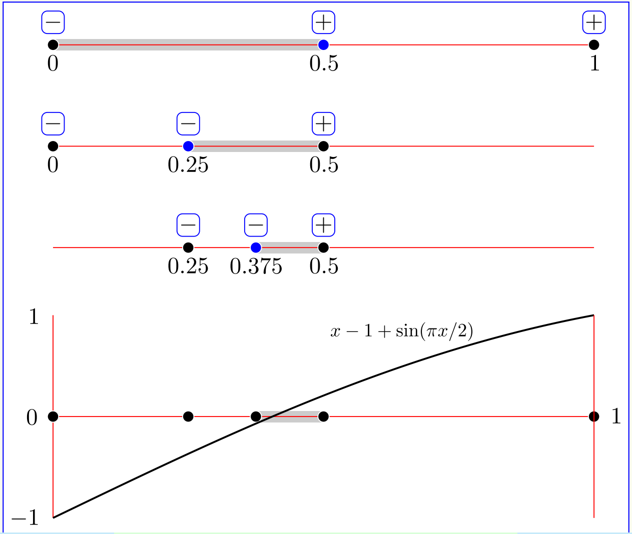
- So we now start with $a = 0.375, b = 0.5$ where $f(0.375) < 0$ and $f(0.5) > 0$.
- Test the point in the middle $x = \dfrac{0.375 + 0.5}{2} = 0.4375$.

$$f(0.4375) = 0.0718932843 > 0.$$

- So our new interval will be $[0.375, 0.4375]$ since the function is negative at $x = 0.375$ and positive at $x = 0.4375$.

So without much work we know the location of a zero inside a range of length $0.0625 = 2^{-4}$. Each iteration will halve the length of the range and we keep going until we reach the precision we need, though it is much easier to program a computer to do it.

Bisection Method Summary Illustration



Definition of Derivative

Definition 5.1 (Derivative as a Function).

- Let $f(x)$ be a function. The **derivative** of $f(x)$ with respect to x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  provided the limit exists.
- If the derivative $f'(x)$ exists for all $x \in (a, b)$ we say that f is **differentiable** on (a, b).
- We can simply write "f is differentiable" to mean "f is differentiable on an interval we are interested in" or "f is differentiable everywhere".

Example 5.2

Example 5.2 (The derivative of $f(x) = 1/x$).

- Compute the limit as follows:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && \text{(the definition)} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{x+h} - \frac{1}{x}\right] && \text{(substituted in the function)} \\
&= \lim_{h\to 0} \frac{1}{h}\,\frac{x-(x+h)}{x(x+h)} && \text{(wrote over a common denominator)} \\
&= \lim_{h\to 0} \frac{1}{h}\,\frac{-h}{x(x+h)} && \text{(started cleanup)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} = \boxed{-\frac{1}{x^2}}.
\end{aligned}
$$

Example 5.3

Example 5.3 (The derivative of $f(x) = \sqrt{x}$).

Compute the derivative, $f'(a)$, of the function $f(x) = \sqrt{x}$ at the point $x = a$ for any $a > 0$.

- Start with the definition of derivative and go from there:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}.$$

- Now, apply the trick of “multiplication by the **conjugate**”:

$$\frac{\sqrt{x} - \sqrt{a}}{x - a} = \frac{\sqrt{x} - \sqrt{a}}{x - a} \times \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} \quad \left(\text{multiplication by } 1 = \frac{\text{conjugate}}{\text{conjugate}}\right)$$

$$= \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} = \frac{1}{\sqrt{x} + \sqrt{a}}.$$

Example 5.3 (Cont'd)

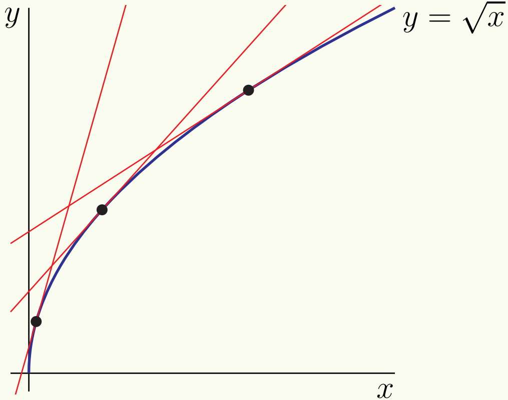
Example 5.3 (Cont'd)

Once we know that $\frac{\sqrt{x}-\sqrt{a}}{x-a} = \frac{1}{\sqrt{x}+\sqrt{a}}$, we can take the limit:

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} \\ &= \lim_{x \to a} \frac{1}{\sqrt{x}+\sqrt{a}} \\ &= \boxed{\frac{1}{2\sqrt{a}}}. \end{aligned}$$

Example 5.3 (Cont'd)

- Three tangents at three points
- With parameter a, the slope of the **tangent** $= \frac{1}{2\sqrt{a}}$.



Notations

- The following notations are all used for "the derivative of $f(x)$ with respect to x":

$$f'(x) \qquad \frac{\mathrm{d}f}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x} f(x),$$

- The following notations are all used for "the derivative of $f(x)$ at $x = a$":

$$f'(a) \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(a) \qquad \frac{\mathrm{d}}{\mathrm{d}x} f(x)\bigg|_{x=a}.$$

Product Rule $(f\,g)' = f'\,g + f\,g'$

By definition, $(f\,g)' = \lim_{h\to 0} \dfrac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$.

$$
\begin{aligned}
(f\,g)' &= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)\,g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)\,(g(x+h) - g(x))}{h} + \lim_{h\to 0} \frac{g(x)\,(f(x+h) - f(x))}{h} \\
&= \lim_{h\to 0} f(x+h)\,\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\,\frac{f(x+h) - f(x)}{h} \\
&= \left(\lim_{h\to 0} f(x+h)\right)\left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right) + \left(\lim_{h\to 0} g(x)\right)\left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right) \\
&= f(x)\,g'(x) + g(x)\,f'(x) \\
&= f'\,g + f\,g'.
\end{aligned}
$$

Quotient Rule $\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$

- By definition,

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0} \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} = \lim_{h\to 0} \frac{1}{h}\, \frac{f(x+h)\,g(x) - f(x)\,g(x+h)}{g(x+h)\,g(x)}$$

$$= \lim_{h\to 0} \frac{1}{h}\, \frac{f(x+h)\,g(x) - f(x)\,g(x) + f(x)\,g(x) - f(x)\,g(x+h)}{g(x+h)\,g(x)}$$

$$= \lim_{h\to 0} \frac{1}{g(x+h)\,g(x)} \left(\frac{f(x+h)\,g(x) - f(x)\,g(x)}{h} + \frac{f(x)\,g(x) - f(x)\,g(x+h)}{h} \right)$$

$$= \lim_{h\to 0} \frac{1}{g(x+h)\,g(x)} \left(g(x)\, \frac{f(x+h) - f(x)}{h} - f(x)\, \frac{g(x+h) - g(x)}{h} \right)$$

Quotient Rule $\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$ (Cont'd)

$$\left(\frac{f}{g}\right)' = \frac{1}{\lim\limits_{h\to 0} g(x+h) \lim\limits_{h\to 0} g(x)} \left(\left(\lim_{h\to 0} g(x)\right) \left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right) - \left(\lim_{h\to 0} f(x)\right) \left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right) \right)$$

$$= \frac{1}{g(x)\,g(x)} \left(g(x)\,f'(x) - f(x)\,g'(x)\right)$$

$$= \frac{f'\,g - f\,g'}{g^2}$$

Chain Rule

Theorem 6.1 (Chain Rule).

Suppose $f(x)$ and $g(x)$ are both differentiable functions. We define

$$F(x) := (f \circ g)(x) = f\big(g(x)\big).$$

Then the derivative of $F(x)$ is

$$F'(x) = f'(g(x)) \; g'(x).$$

- Let $y := f(x)$ and $u := g(x)$. We can write the theorem as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Proof of Chain Rule (1)

- By definition, $u'(x) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}$.
- Let us define,

$$v(h) = \begin{cases} \frac{u(x+h) - u(x)}{h} - u'(x) & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$$

and notice that $\lim_{h \to 0} v(h) = 0 = v(0)$ and so $v(h)$ is continuous at $h = 0$.

- Suppose $h \neq 0$, we rewrite $v(h)$ to get,

$$u(x+h) = u(x) + h\left(v(h) + u'(x)\right) \tag{1}$$

Proof of Chain Rule (2)

- Since $f(x)$ is differentiable, we can do something similar.

$$w(k) = \begin{cases} \dfrac{f(z+k) - f(z)}{k} - f'(z) & \text{if } k \neq 0 \\ 0 & \text{if } k = 0 \end{cases}$$

- By the same argument, $w(k)$ is continuous at $k = 0$.

$$f(z+k) = f(z) + k\left(w(k) + f'(z)\right). \tag{2}$$

Proof of Chain Rule (3)

- Use the definition of the derivative and evaluate

$$\frac{d}{dx}[f[u(x)]] = \lim_{h\to 0}\frac{f[u(x+h)] - f[u(x)]}{h} \tag{3}$$

- By (1), the numerator becomes,

$$f[u(x+h)] - f[u(x)] = f\left[u(x) + h\left(v(h) + u'(x)\right)\right] - f[u(x)].$$

- If we then define $z := u(x)$ and $k := h\left(v(h) + u'(x)\right)$, we can use (2) to further write this as

$$\begin{aligned} f[u(x+h)] - f[u(x)] &= f\left[u(x) + h\left(v(h) + u'(x)\right)\right] - f[u(x)] \\ &= f[u(x)] + h\left(v(h) + u'(x)\right)\left(w(k) + f'[u(x)]\right) - f[u(x)] \\ &= h\left(v(h) + u'(x)\right)\left(w(k) + f'[u(x)]\right). \end{aligned}$$

Proof of Chain Rule (4)

- Plugging this into (3) gives,

$$\begin{aligned}\frac{d}{dx}\left[f\left[u\left(x\right)\right]\right] &= \lim_{h\to 0}\frac{h\left(v\left(h\right)+u'\left(x\right)\right)\left(w\left(k\right)+f'\left[u\left(x\right)\right]\right)}{h}\\ &= \lim_{h\to 0}\left(v\left(h\right)+u'\left(x\right)\right)\left(w\left(k\right)+f'\left[u\left(x\right)\right]\right).\end{aligned}$$

- Next, recall that $k = h\left(v\left(h\right)+u'\left(x\right)\right)$, so $\lim_{h\to 0} k = 0$.
- It follows that $\lim_{h\to 0} w\left(k\right) = w\left(\lim_{h\to 0} k\right) = w\left(0\right) = 0$.
- Therefore,

$$\begin{aligned}\frac{d}{dx}\left[f\left[u\left(x\right)\right]\right] &= \lim_{h\to 0}\left(v\left(h\right)+u'\left(x\right)\right)\left(w\left(k\right)+f'\left[u\left(x\right)\right]\right) = u'\left(x\right)f'\left[u\left(x\right)\right]\\ &= f'\left[u\left(x\right)\right]\frac{du}{dx} = f'\big(g(x)\big)g'(x).\end{aligned}$$

Example of Chain Rule

- Suppose $h(x) = \sqrt{4x + 1/x}$. It is a composite of $f(u) = \sqrt{u}$ and $g(x) = 4x + 1/x$.
  - Step 1: $$\frac{\mathrm{d}}{\mathrm{d}u}\sqrt{u} = \frac{1}{2}\frac{1}{\sqrt{u}}.$$
  - Step 2: $$\left(\frac{\mathrm{d}}{\mathrm{d}u}\sqrt{u}\right)\bigg|_{u=g(x)} = \frac{1}{2}\frac{1}{\sqrt{4x+1/x}}.$$
  - Step 3: $$\frac{\mathrm{d}}{\mathrm{d}x}g(x) = 4 - \frac{1}{x^2}.$$
  - Step 4: $$\frac{\mathrm{d}}{\mathrm{d}x}h(x) = \frac{1}{2}\frac{1}{\sqrt{4x+1/x}}\left(4 - \frac{1}{x^2}\right).$$

Standard Forms

Derivatives	Antiderivatives
$\dfrac{d}{dx}((x+a)^n) = n(x+a)^{n-1}$	$\displaystyle\int (x+a)^n dx = \frac{1}{n+1}(x+a)^{n+1} + C, \quad n \neq 1$
$\dfrac{d}{dx}(e^{x+a}) = e^{x+a}$	$\displaystyle\int e^{x+a} dx = e^{x+a} + C$
$\dfrac{d}{dx}(b^{x+a}) = \ln(b) \cdot b^{x+a}, \quad b > 0$	$\displaystyle\int b^{x+a} dx = \frac{b^{x+a}}{\ln(b)} + C, \quad b > 0, b \neq 1$
$\dfrac{d}{dx}(\ln(x+a)) = \dfrac{1}{x+a}$	$\displaystyle\int \frac{1}{x+a} dx = \ln\lvert x+a \rvert + C$
$\dfrac{d}{dx}(\sin(x+a)) = \cos(x+a)$	$\displaystyle\int \sin(x+a)\, dx = -\cos(x+a) + C$
$\dfrac{d}{dx}(\cos(x+a)) = -\sin(x+a)$	$\displaystyle\int \cos(x+a)\, dx = \sin(x+a) + C$

A Nice Example

- Find $\lim_{x\to 1} \frac{\log_4(x+1) - 0.5}{x-1}$.
- Consider the function $f(x) = \log_4(x+1)$. Since $f(1) = 0.5$, the limit is a derivative when $x \to 1$. So by the definition of derivative,

$$\lim_{x\to 1} \frac{f(x) - f(1)}{x-1} = \lim_{x\to 1} \frac{\log_4(x+1) - 0.5}{x-1} = f'(1).$$

- We cannot use $(\ln(x))' = 1/x$ because the base of $f(x)$ is 4 rather than e.
- By the law of the change of logarithmic base, we have

$$f(x) = \log_4(x+1) = \frac{\ln(x+1)}{\ln 4} \implies f'(x) = \frac{1}{(x+1)\ln(4)} \implies f'(1) = \boxed{\frac{1}{2\ln(4)}}.$$

Keywords