Section 1
Overview

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Inspiring Quote

Be a free thinker and don't accept everything you hear as truth. Be critical and evaluate what you believe in.

— Aristotle

Main Goals of Practical English (PE) II

- Main goal of PE 2—revise calculus, but in English.
- Concrete goal of PE2 A—develop critical thinking skill.
- Concrete goal of PE2 B—gain exposure and experience in different application domains.
- Eventual goal—develop the capability to read research papers and monographs related to informatics and data science.

Lesson Format

- Assumption: You have done Calculus (微分積分) in Japanese.
- The lesson is conducted online from 12:50 PM to 2:35 PM every Friday from June 6 onward.
- Each lesson will be followed by in-class exercise from 2:35 PM to 4:05 PM.
- All the learning materials are available for download on Hirodai mooddle.
- All the assignments are to be submitted to Hirodai mooddle.
- Make sure you don't submit an empty Excel file.

In-Class Exercise (ICE) 100%

- Each lesson comes with an ICE to help you stay alert and focused.
- Use the Excel template provided to fill in your name (Eastern style in Kanji) and your student ID.
- Submit your completed Excel file to Hirodai moodle.
- Verify your submitted Excel file by opening it on Hirodai moodle.
- The deadline of ICE submission is 23:59 Hours Friday.

Name	石破茂
Student ID	buvwxyz
ICE Q1	A
ICE Q2	B
ICE Q3	A
ICE Q4	C
ICE Q5	D
ICE Q6	C
ICE Q7	A
ICE Q8	B
ICE Q9	B
ICE Q10	D

Will be graded by a Python code

Set

Definition 3.1 (Set).

A **set** is a well-defined collection of objects, which are called the '**elements**' of the set. Here, 'well-defined' means that it is possible to determine if something **belongs to** the collection or not, without prejudice.

Definition 3.2 (Notation for set inclusion and exclusion).

Let A be a set.

- If x is an element of A then we write $x \in A$ (**inclusion**), which is read as 'x is in A'.
- If x is *not* an element of A then we write $x \notin A$ (**exclusion**), which is read as 'x is not in A'.

Subset, Empty Set, and Set Operations

Definition 3.3 (Subset).

Given sets A and B, we say that the set A is a **subset** of the set B and write '$A \subseteq B$' if every element in A is also an element of B.

Definition 3.4 (Empty set).

The **empty set** $\emptyset$ is the set which contains no element. That is,

$$\emptyset = \{\,\} = \{x \,|\, x \neq x\}.$$

Definition 3.5 (Intersection and union).

Suppose A and B are sets.

- The **intersection** of A and B is $A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}$
- The **union** of A and B is $A \cup B = \{x \,|\, x \in A \text{ or } x \in B \text{ (or both)}\}$

Integers

Definition 3.6 (Integers).

The simplest numbers are the *positive integers*

$$1, 2, 3, 4, \cdots$$

the number *zero*

$$0,$$

and the *negative integers*

$$\cdots, -4, -3, -2, -1.$$

Together these form the **integers** or "**whole numbers**."

- The symbol for the set of all integers is $\mathbb{Z}$.
- Strictly positive integers: $\mathbb{Z}^+ := \{1, 2, 3, \ldots\}$.

Rational Numbers

Definition 3.7 (Rational numbers).

A **rational number**, also called a **fraction**, is formed by dividing one whole number called the **numerator** by another nonzero whole number called the **denominator**.

- Example

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{3}, \ldots$$

and

$$-\frac{1}{2}, -\frac{1}{3}, -\frac{2}{3}, -\frac{1}{4}, -\frac{2}{4}, -\frac{3}{4}, -\frac{4}{3}, \ldots$$

- In particular, zero is a rational number: $\frac{0}{d}$ for any non-zero integer d in $\mathbb{Z}$.
- The symbol for the set of all rational numbers is $\mathbb{Q}$.
- By definition, any whole number is a rational number: $\mathbb{Z} \subset \mathbb{Q}$.

Some Unusual Whole Numbers

- Fine structure constant $\alpha = \dfrac{e^2}{2\epsilon_0 hc} \approxeq \dfrac{1}{137}$.
  - e is the electric charge (:= 1.602176634$\times 10^{-19}$ C).
  - ϵ_0 is the electric constant (or permittivity) in vacuum (=8.85418782$\times 10^{-12}$ C$\cdot$ V$^{-1}\cdot$ m^{-1})
  - h is the Planck constant (:= 6.62607015$\times 10^{-34}$ J·s)
  - c is the speed of light in vacuum (= 299792458 m/s)
- All the 1-digit and 2-digit combinations of 137 are prime numbers:
  - 3, 7
  - 13, 17
  - 31, 37
  - 71, 73
- **Pythagorean primes**
  - $37 = 6^2 + 1^2$ $73 = 8^2 + 3^2$ $137 = 11^2 + 4^2$

$\sqrt{2}$ Is Not a Rational Number

- Show that the equation $p^2 = 2$ is not satisfied by any rational p.
- Proof by contradiction:
  - If there were such a p, we could write $p = a/b$ where a and b are integers that are not both even.
  - Then we have
  $$a^2 = 2b^2.$$
  - Hence a^2 is even, which means that a is even too. (if a were odd, a^2 would be odd.)
  - So let $a = 2k$. Then we obtain
  $$b^2 = 2k^2$$
  - It means that b^2 must be even, which implies that b must be even too.
  - So the assumption leads to a contradiction.
  - Therefore, we have proved that p cannot be a rational number. □

Transcendental Numbers

- When a number that is not algebraic—that is, not a root (i.e., solution) of a nonzero polynomial equation with integer coefficients.
- $\pi \approx$ 3.1416 and $\mathrm{e} \approx 2.7813$ are transcendental.
- The two mathematical constants are crucially important in science and engineering.
- Most beautiful formula in mathematics:

$$\mathrm{e}^{\mathrm{i}\pi} + 1 = 0.$$

where $\mathrm{i} := \sqrt{-1}$.

An Application

- Why should we learn **functions**, **derivatives**, and **integrals**?
- Besides training our mind to become more logical in thinking, learning these mathematical constructs allows us to do some fun things.
- The number π is defined as the ratio of the **circumference** of a **circle** over its **diameter**.
- How do we know that π is an irrational number? We can apply calculus to prove it.
- In particular, the following rules of calculus shall be applied.

$$\frac{d}{dx}cx^n = ncx^{n-1} \qquad \frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}f(x)g(x) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) \qquad \int_0^\pi \frac{dF}{dx}\,dx = F(\pi) - F(0).$$

π is irrational

- We shall prove by contradiction. Suppose $\pi = a/b$ for non-zero integers a and b.
- Define $f(x) = \dfrac{x^n(a-bx)^n}{n!}$ and a combination of the derivatives of $f(x)$ for any positive integer n:

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - ... + (-1)^n f^{(2n)}(x).$$

- For $i < n$, all the derivatives $f^{(i)}(x)$ contain the terms $Cx^g(a-bx)^h$, where C is a coefficient. The integers g and h add up to a value less than $2n$.
- For $i = n$, the derivative $f^{(n)}(x)$ has a term where x^n is differentiated n times, which cancels away $n!$, leaving behind $(a-bx)^n$.
- The derivative $f^{(n)}(x)$ also contains another term where $(a-bx)^n$ is differentiated n times, leaving behind $(-b)^n x^n$.

π is irrational (cont'd)

- The rest of $f^{(n)}(x)$ is of the form $Cx^g(a-bx)^h$, where C is an integer, $g>0$, $h>0$, and $g+h=n$.
- At $x=0$, only the $(a-bx)^n$ term of $f^{(n)}(x)$ is non-zero (specifically $f^{(n)}(0)=a^n$).
- At $x=\pi=a/b$, only the $(-b)^n x^n$ term of $f^{(n)}(x)$ is non-zero (specifically $f^{(n)}(\pi)=(-1)^n a^n$).
- Consequently, each $f^{(n)}(x)$ has integral values for $x=0$ and also for $x=\pi=a/b$.
- For $i>n$, it is easy to see that all the derivatives $f^{(n)}(x)$ contain the terms Ax^k and $B(a-bx)^l$, where A, B, k, and l are all integers.
- Thus, $F(\pi)$ and $F(0)$ must be integers.

π is irrational (cont'd)

- Now, $F''(x) = f^{(2)}(x) - f^{(4)}(x) - \ldots + (-1)^n f^{(2n+2)}(x)$.
- Obviously, for any **power function** $p(x) = x^m$, it must be that $p^{(r)}(x) = 0$ when $r > m$.
- So $(-1)^n f^{(2n+2)}(x) = 0$, since $f(x)$ is a **polynomial** of **order** $2n$.
- Therefore $F''(x) = f^{(2)}(x) - f^{(4)}(x) - \ldots + (-1)^n f^{(2n)}(x)$. It follows that

$$\frac{d}{dx}\Big[F'(x)\sin x - F(x)\cos x\Big] = F''(x)\sin x + F(x)\sin x = f(x)\sin x.$$

- Upon integration from 0 to π, we obtain

$$\int_0^\pi f(x)\sin x\,dx = \Big[F'(x)\sin x - F(x)\cos x\Big]_0^\pi = F(\pi) + F(0) \neq 0.$$

π is irrational (cont'd)

- For $0 < x < \pi$, $\sin x \leqq 1$ and $f(x) > 0$.
- For the numerator of $f(x)$, i.e., $x^n(a - bx)^n$, we let the first x of x^n equal to π and the second x in $(a - bx)^n$ equal to zero.
- Therefore, for $0 < x < \pi$,

$$0 < f(x)\sin x < \frac{\pi^n a^n}{n!}.$$

- Integrate each term of the inequality from 0 to π, we obtain

$$0 < \int_0^\pi f(x)\sin x\,dx < \frac{\pi^{n+1}a^n}{n!}.$$

π is irrational (cont'd)

- Since $\int_0^\pi f(x)\sin x\,dx = F(\pi) + F(0) \in \mathbb{Z} \setminus \{0\}$, we have

$$0 < F(\pi) + F(0) < \frac{\pi^{n+1}a^n}{n!}.$$

- When n approaches infinity, we have a non-zero integer equals to zero.
- So there is a contradiction.
- Since $f(x)$ and $F(x)$ are well-defined functions for $0 < x < \pi$, it must be that the starting assumption $\pi = a/b$ is invalid.
- Hence, π cannot be a rational number. □

What is infinity?

- Extremely humongous number
  - Googol is 10^{100}.
  - Googolplex is 10 billion Googols.
  - Googolplexian is Googol^{1000}
- **Infinity** is not a real number.
- Infinity is boundless, endless.
- For any real number x, $-\infty < x < \infty$.
- Infinity does not change.

Changeless and Endless

Infinity arithmetic

- $\infty + \infty = \infty$
- $-\infty + (-\infty) = -\infty$
- $\infty \times \infty = \infty$
- $-\infty \times (-\infty) = \infty$
- $-\infty \times \infty = -\infty$

For any $-\infty < x < \infty$,

- $x + \infty = \infty$
- $x + (-\infty) = -\infty$
- $x - \infty = -\infty$
- $x - (-\infty) = \infty$
- $x^+ \times \infty = \infty$
- $x^+ \times (-\infty) = -\infty$
- $x^- \times \infty = -\infty$
- $x^- \times (-\infty) = \infty$

Undefined operations (UDO)

- $0 \times \infty$
- $0 \times (-\infty)$
- $\infty + (-\infty)$
- $\infty - \infty$
- $\frac{\infty}{\infty}$
- ∞^0
- 1^∞

Order

Definition 5.1 (Order).

Let S be a set. An **order** on S is a relation, denoted by $<$, with the following two properties:

(i) If $x \in S$ and $y \in S$, then one and only one of the statements is true:

$$x < y, \quad x = y, \quad y < x.$$

(ii) If $x, y, z \in S$, and if $x < y$ and $y < z$, then $x < z$.

Definition 5.2 (Ordered Set).

An **ordered set** is a set S in which an order is defined.

For example, $\mathbb{Q}$ is an ordered set if $r < s$ is defined to mean that $s - r$ is a positive rational number.

Upper Bound and Supremum

Definition 5.3 (Upper Bound).

Suppose S is an ordered set. For a subset $E \subset S$, if there exists a $\delta \in S$ such that for every $x \in E$, $x \leq \delta$, we say that E is **bounded above**, and call δ an **upper bound** of E.

Definition 5.4 (Supremum).

Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

(i) α is an upper bound of E.

(ii) If $\gamma < \alpha$, then γ is not an upper bound of E.

Then α is called the **least upper bound** of E or the **supremum** of E, and we write

$$\alpha = \sup E.$$

Lower Bound and Infimum

Definition 5.5 (Lower Bound).

Suppose S is an ordered set. For a subset $E \subset S$, if there exists a $\delta \in S$ such that for every $x \in E$, $x \geqq \delta$, we say that E is **bounded below**, and call δ a **lower bound** of E.

Definition 5.6 (Infimum).

Suppose S is an ordered set, $E \subset S$, and E is bounded below. Suppose there exists an $\alpha \in S$ with the following properties:

(i) α is a lower bound of E.

(ii) If $\alpha < \gamma$, then γ is not a lower bound of E.

Then α is called the **greatest lower bound** of E or the **infimum** of E, and we write

$$\alpha = \inf E.$$

Example (a)

- Let E consist of all numbers $\dfrac{1}{n}$, where $n = 1, 2, 3, \ldots$.

- Then $\sup \dfrac{1}{n} = 1$, which is in E, and $\inf E = 0$, which is not in E.

Example (b)

- Let A be the set of all positive rationals p such that $p^2 < 2$ and let B consist of all positive rationals p such that $p^2 > 2$.
- A and B are subsets of the ordered set $\mathbb{Q}$.
- The set A is bounded above. The upper bounds are exactly the members of B.
- Since B contains no smallest member, A has no least upper bound in $\mathbb{Q}$. That is

$$\sup A \text{ does not exist in } \mathbb{Q}.$$

What is function?

Definition 6.1 (Function).

Let A, B be **non-empty sets**. A **function** f from A to B is a rule or formula that takes elements of A as **inputs** and returns elements of B as **outputs**. We write this as

$$f : A \to B.$$

If f takes $a \in A$ as an input and returns $b \in B$, then we write

$$f(a) = b.$$

Every function must satisfy the following two conditions:

- It is defined on every possible input from the set A. No matter which element $a \in A$ we choose, the function must return an element $b \in B$ so that $f(a) = b$.
- It returns one result only for each input. So if $f(a) = b_1$ and $f(a) = b_2$, then the only way that f can be a function is if $b_1 = b_2$.

Sets of a Function

Definition 6.2 (Domain, Codomain, Image, Range).

Let $f : A \to B$ be a function. Then

- the set A of inputs to our function is the **domain** of f,
- the set B which contains all the results is called the **codomain**,
- We read "$f(a) = b$" as "f of a is b", but sometimes we might say "f maps a to b" or "b is the **image** of a".
- The codomain B must contain all outputs of the function, but it might also contain a few other elements. The subset of B that is exactly the outputs of A is called the **range** of f, i.e.,

$$\begin{aligned} \text{range of } f &= \{b \in B \mid \text{there is some } a \in A \text{ so that } f(a) = b\} \\ &= \{f(a) \in B \mid a \in A\}. \end{aligned}$$

  The only elements allowed in the range are those elements of B that are the images of elements in A.

Two Simple Examples

(A) Let $h : [0, \infty) \to [0, \infty)$ be defined by the formula $h(x) = \sqrt{x}$.

- Then the domain and codomain are both the set $[0, \infty)$.
- In this example, the range is equal to the codomain, namely $[0, \infty)$.

(B) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by the formula $g(x) = x^2$.

- Then the domain and codomain are both the set of all real numbers.
- But the range is the set $[0, \infty)$.

One-to-one (injective) and Horizontal Line Test

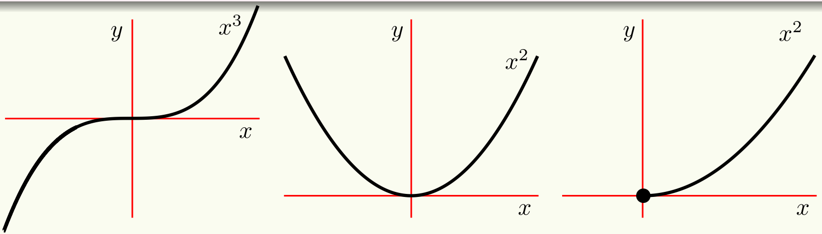
Definition 6.3 (Injective).

A function f is **one-to-one** (**injective**) when it never takes the same y value more than once. That is

$$\text{if } x_1 \neq x_2 \text{ then } f(x_1) \neq f(x_2)$$

Definition 6.4 (Horizontal Line Test).

A function is one-to-one if and only if no horizontal line $y = c$ intersects the graph $y = f(x)$ more than once.



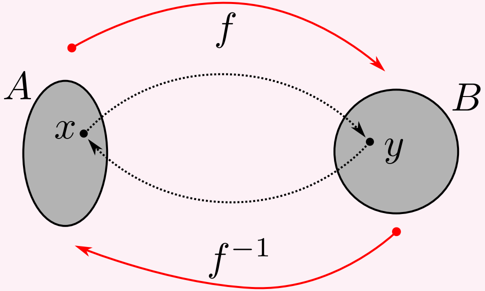
Inverse Function

Definition 6.5 (Inverse).

Let f be a **one-to-one** function with **domain** A and **range** B. Then its **inverse function** is denoted f^{-1} and has **domain** B and **range** A. It is defined by

$$f^{-1}(y) = x \qquad \text{whenever} \qquad f(x) = y$$

for any $y \in B$.



Example of Inverse Function

Let $f(x) = x^5 + 3$ on domain $\mathbb{R}$. To find its inverse we do the following

- Write $y = f(x)$; that is $y = x^5 + 3$.
- Solve for x in terms of y: $x^5 = y - 3$, so $x = (y-3)^{1/5}$.
- The solution is $f^{-1}(y) = (y-3)^{1/5}$.
- Recall that the "y" in $f^{-1}(y)$ is a **dummy variable**. That is, $f^{-1}(y) = (y-3)^{1/5}$ means that if you feed the number y into the function f^{-1} it outputs the number $(y-3)^{1/5}$.
- You may call the input variable anything you like. So if you wish to call the input variable "x" instead of "y" then just replace every y in $f^{-1}(y)$ with an x. That is, $f^{-1}(x) = (x-3)^{1/5}$.

Keywords