§ 2.4 Standard Probability Space

Suppose (Ω_1, P_1) and (Ω_2, P_2) are probability spaces. The bijection from Ω_1 into Ω_2, namely, when the one-to-one mapping from Ω_1 onto Ω_2 satisfies

$$A_1 \in \mathscr{D}(P_1) \Longleftrightarrow f(A_1) \in \mathscr{D}(P_2) \Longrightarrow P_1(A_1) = P_2\big(f(A_1)\big)\,,$$

the mapping f from (Ω_1, P_1) into (Ω_2, P_2) is called the **strong isomorphism mapping**. When such mapping f exists, (Ω_1, P_1) is said to be in **strong isomorphism** with (Ω_2, P_2), and is expressed as

$$(\Omega_1, P_1) \approx (\Omega_2, P_2)\,.$$

When f is to be referred to, (Ω_1, P_1) is in strong morphism with (Ω_2, P_2) by f, and is written as

$$(\Omega_1, P_1) \approx (\Omega_2, P_2) \quad (f)$$

Obviously, since

Reflective law $\quad (\Omega, P) \approx (\Omega, P) \quad (I), I$ is an identity mapping,

Symmetric law $\quad (\Omega_1, P_1) \approx (\Omega_2, P_2) \quad (f) \Longrightarrow (\Omega_2, P_2) \approx (\Omega_1, P_1) \quad (f^{-1})$

Transitive law $\quad (\Omega_1, P_1) \approx (\Omega_2, P_2) \quad (f)\,, \qquad (\Omega_2, P_2) \approx (\Omega_3, P_3) \quad (g) \Longrightarrow (\Omega_1, P_1) \approx (\Omega_3, P_3) \quad (g \circ f)$

are valid, strong morphism and equivalence are of the same type.

A slightly weaker concept than strong isomorphism is **isomorphism**. Let Ω' be a subset of the probability space (Ω, P) with P measure of 1. The **restriction** of P into Ω', $P' = P|_{\Omega'}$ is defined as

$$\mathscr{D}(P') = \big\{A' \in \mathscr{D}(P) \big| A' \subset \Omega'\big\}\,,$$
$$P'(A') = P(A')\,, \qquad A' \in \mathscr{D}(P')$$

Clearly, P' is a probability measure on Ω'. The probability space (Ω', P') is called the **restriction** of (Ω, P) toward Ω'. (Ω_1, P_1) and (Ω_2, P_2) are said to be **isomorphic** when the subset $\Omega'_i \quad (i = 1, 2)$ of (Ω_i, P_i) with P_i measure of 1 exists, such that the restriction of (Ω_1, P_1) toward Ω'_1 and the restriction of (Ω_2, P_2) toward Ω'_2 are strongly isomorphic. It is expressed as

$$(\Omega_1, P_1) \sim (\Omega_2, P_2)\,.$$

Isomorphism is also of the same type as equivalence. As reflective law and symmetric law are obvious, only the transitive law is proven. Suppose

$$(\Omega_1, P_1) \sim (\Omega_2, P_2)\,, \qquad (\Omega_2, P_2 \sim (\Omega_3, P_3)\,.$$

Suppose the restrictions (Ω'_1, P'_1) of (Ω_1, P_1), (Ω'_1, P'_1) and (Ω''_2, P''_2) of (Ω_2, P_2), as well as (Ω'_3, P'_3) of (Ω_3, P_3) exit, such that

$$(\Omega'_1, P'_1) \approx (\Omega'_2, P'_2) \quad (f)\,, \qquad (\Omega''_2, P''_2) \sim (\Omega'_3, P'_3) \quad (g)\,.$$

Let

$$\widetilde{\Omega}_2 = \Omega_2' \cap \Omega_2'' , \qquad \widetilde{\Omega}_1 = f^{-1}\big(\widetilde{\Omega_2}\big) , \qquad \widetilde{\Omega}_3 = g\big(\widetilde{\Omega_2}\big) ,$$

and $P_i\big(\widetilde{\Omega}_i\big) = 1 \ \ (i = 1, 2, 3)$. Denote the restriction of P_i to $\widetilde{\Omega}_i$ by $\widetilde{P}_i$. Then the restriction of (Ω_i, P_i) is $(\widetilde{\Omega}_i, \widetilde{P}_i)$. Moreover, since

$$(\widetilde{\Omega}_1, \widetilde{P}_1) \approx (\widetilde{\Omega}_2, \widetilde{P}_2) \approx (\widetilde{\Omega}_3, \widetilde{P}_3) ,$$

$(\Omega_1, P_1) \sim (\Omega_3, P_3)$ is obtained.

Definition 2.1 Let P be the complete probability measure on Ω. When (Ω, P) is isomorphic to $(\boldsymbol{R}^1, \mu)$ (μ is a regular probability measure on $\boldsymbol{R}^1$.), (Ω, P) is called the **standard probability space**, and P is the **standard probability measure** on Ω.

As shown below, probability spaces that are useful for application are mostly standard probability spaces.

Theorem 2.5 When Ω is countable (finite or infinite), the probability space (Ω, P) obtained from endowing Ω with probability measure P on the domain 2^{Ω} is a standard probability space.

Proof Prove the case for which Ω is countably infinite. Let the elements of Ω be $\omega_1, \omega_2, \cdots$. Denote the regular probability measure on $\boldsymbol{R}^1$ by Q:

$$Q(E) = \sum_{\omega_n \in E} P\{\omega_n\} .$$

Note that $Q(\boldsymbol{N}) = 1$, where $\boldsymbol{N}$ is the set of natural numbers. Let the restriction to $\boldsymbol{N}$ be Q'. From the correspondence $n \longrightarrow \omega_n$, since $(\boldsymbol{N}, Q')$ is strongly isomorphic to (Ω, P), $(\boldsymbol{R}, Q)$ is isomorphic to (Ω, P). Accordingly, (Ω, P) is standard. ∎

The regular probability measure on $\boldsymbol{R}^1$ is standard. All the regular probability measures on $\boldsymbol{R}^n$ ($n = 2, 3, \cdots$) are also standard. More generally, the regular probability measure on a complete separable metric space is standard as well. For the proof of this statement, some preparation is needed.

Lemma 2.1 Let P be the regular probability measure on a complete separable metric space S. For any $A \in \mathscr{D}(P)$ and any $\epsilon > 0$, take an appropriate compact set $K \subset A$, it is possible to have

$$P(A - K) < \epsilon .$$

Note When P is the regular probability measure on the topological space S. A with the above mentioned properties is call ***K* regular**. P such that for all P measurable sets that are ***K* regular** is called ***K* regular**. With these terminologies, the above lemma can be expressed as

"The probability measure on a complete separable metric space is K compact."

Proof of Lemma 2.1 Let ρ be a metric on S. First, consider $A = S$. Since S is separable, there are dense countable sets $\{a_1, a_2, \cdots\}$ in S. For any n, since

$$S = \bigcup_{n=1}^{\infty} B_{nk} , \qquad B_{nk} = \left\{ x \in S \,\middle|\, \rho(x, a_n) \leqq \frac{1}{k} \right\} \quad \text{(a closed ball with center } a_n \text{ and radius } 1/k)$$

$$\bigcup_{n=1}^{N} B_{nk} \uparrow S \quad (N \to \infty)$$

for each k, make $N(k)$ sufficiently large, and it is possible to have

$$P(S - B_k) < 2^{-(k+1)}\epsilon\,, \qquad B_k = \bigcup_{k=1}^{N(k)} B_{nk}\,.$$

As B_k are all closed sets,

$$K = \bigcap_{n=1}^{\infty} B_k$$

is also a closed set. Again, for all k,

$$K \subset B_k = \bigcup_{n=1}^{N(k)} B_{nk}\,,$$

and as B_{nk}'s radius is below $2/k$, K is bounded. Since S is complete, the bounded subset K is compact. Moreover

$$P(S-K) = P\left(S - \bigcap_k B_k\right) = P\left(\bigcup_k (S - B_k)\right) \leqq \sum_k P(S - B_k) < \epsilon\,.$$

Thus, S is understood to be K regular.

Next, let A be a generic Borel set. Then for any $\epsilon > 0$, take a closed set $F \subset A$ and an open set $G \supset A$, show that

$$P(G - F) < \epsilon\,.$$

For this purpose, one just need to say that the totality $\mathscr{B}$ of all A that has this property contains the Borel set family $\mathscr{B}(S)$. As any arbitrary open set in the metric space is a countable union of closed sets, it is obvious that $\mathscr{B}$ contains all open sets. To see that $\mathscr{B}$ is closed under set complement, it is obvious that from $G \supset A \supset F$, $G^c \subset A^c \subset F^c$ is obtained. Let $A_n \in \mathscr{B}$ $(n = 1, 2, \cdots)$, take a sequence $\{F_n\}$ of closed sets and a sequence $\{G_n\}$ of open sets such that

$$F_n \subset A_n \subset G_n\,, \qquad P(G_n - F_n) < 2^{-n}\epsilon\,.$$

Then

$$\bigcup_n F_n \subset \bigcup_n A_n \subset \bigcup_n G_n\,,$$
$$P\left(\bigcup_n G_n - \bigcup_n F_n\right) \leqq \sum_n P(G_n - F_n) < \epsilon \qquad \text{Exercise 2.1 (iv)}\,.$$

Accordingly, when N is sufficiently large,

$$P\left(\bigcup_n G_n - \bigcup_{n=1}^{N} F_n\right) \leqq \epsilon\,, \qquad \bigcup_{n=1}^{N} F_n \subset A \subset \bigcup_n G_n$$

is obtained, and it means $A \in\subset B$. Accordingly, $\mathscr{B}$ is closed under countable union. It follows that $\mathscr{B}$ is a σ additive family that contains the family of open sets. Consequently, $\mathscr{B} \supset \mathscr{B}(S)$.

From the above, for any Borel set A and any $\epsilon > 0$, there exists a closed set $F \subset A$ such that

$$P(A - F) < \epsilon \,.$$

As P is regular, it also holds with respect to all P measurable sets. If a compact set K introduced by S is first taken, then $F \cap K$ is a compact subset of A, and

$$P(A - F \cap K) \leqq P(A - F) + P(A - K) < 2\epsilon \,.$$

As ϵ is any positive number, this shows that A is K regular. Accordingly, P is K regular. ▮

Lemma 2.2 (**Lusin's Theorem**) Suppose S is a complete separable metric space, T is a separable metric space, P is a regular probability measure on S, and $f : S \longrightarrow T$ is P-measurable (i.e. $\mathscr{D}(P)/\mathscr{B}(T)$-measurable). Then for any P-measurable set $A \subset S$ and any $\epsilon > 0$, take an appropriate compact set $K = K(A, \epsilon) \subset A$, and it is possible to have

$$P(A - K) < \epsilon \,, \qquad \text{restricted mapping } f_K \equiv f|_K : S \longrightarrow T \text{ is continuous.}$$

Proof Lusin's theorem is known for the case of $S = T = \boldsymbol{R}^1$. The proof is intrinsically o different from this special case. Denote the metrics on S and T by, respectively, ρ and d. Since T is separable, there is a dense and countable set $\{b_1, b_2, \cdots\} \subset T$. For any k,

$$T = \bigcup_n B_{nk} \,, \qquad B_{nk} = \left\{ y \in T \,\middle|\, d(y, b_n) \leqq \frac{1}{k} \right\} .$$

Now, let

$$A_{nk} = A \cap f^{-1}\big(B_{nk}\big) \,, \qquad A'_{nk} = A_{nk} - \bigcup_{i<n} A_{ik} \,,$$

which is P-measurable. Moreover

$$\begin{aligned} A &= A \cap S = A \cap f^{-1}(T) \\ &= \bigcup_n A \cap f^{-1}\big(B_{nk}\big) = \bigcup_n A_{nk} = \sum_n A'_{nk} \,. \end{aligned}$$

From the previous lemma, there exists a compact subset $K_{nk} \subset A'_{nk}$ such that

$$P\big(A'_{nk} - K_{nk}\big) \leqq 2^{-n-k}\epsilon \,,$$

Accordingly

$$\begin{aligned} P\Big(A - \sum_n K_{nk}\Big) &\leqq \sum_n P\big(A_{nk} - K_{nk}\big) \qquad \text{(Exercise 2.1 (v))} \\ &< 2^{-k}\epsilon \,. \end{aligned}$$

Set $N(k)$ to be sufficiently large, and let $K_k = \sum_{n=1}^{N(k)} K_{nk}$. Then K_k is compact, and

$$P(A - K_k) < 2^{-k}\epsilon \,.$$

Furthermore, let $K = \bigcap_{k=1}^{\infty}$, K too is compact, and

$$P(A - K) \leqq \sum_k P(A - K_k) < \epsilon\,.$$

If the continuity of $f_K = f|_K : K \longrightarrow T$, then the proof of this lemma is obtained. First, from $K \subset K_k$, one has

$$K = K \cap K_k = \sum_{n=1}^{N(k)} K \cap K_{nk}\,.$$

If there are some empty ones amongst $K \cap K_{nk}$, they are omitted from the union. Take any arbitrary a_{nk} in K_{nk}, fix it, and define $g_n : K \longrightarrow T$ as

$$g_n(x) = f\big(a_{nk}\big)\,, \qquad x \in K_{nk}\,, \qquad n = 1, 2, \cdots, N(k)\,.$$

$K \cap K_{nk}$ ($n = 1, 2, \cdots, N(k)$) are mutually exclusive compact sets. As g_n is fixed on every set, g_n is continuous. Also with respect to $x \in K \cap K_{nk}$, since $f(a_{nk}), f(x) \in f(K_{nk}) \subset f(A_{nk}) \subset B_{nk}$,

$$d\big(g_n(x), f(x)\big) = d\big(f(a_{nk}), f(x)\big) \leqq \frac{2}{k}\,, \qquad n = 1, 2, \cdots, N(k)\,.$$

Accordingly, with respect to $x \in K$,

$$d\big(g_n(x), f_K(x)\big) < \frac{2}{k}\,,$$

and $\{g_n\}$ converges uniformly to f_K. Hence, f_K is continuous. ∎

Suppose S and T are general sets, P is a probability measure on S, and $f : S \longrightarrow T$ is any arbitrary mapping. Then define the set function Q on T as

$$\mathscr{D}(Q) = \big\{B \subset T | f^{-1}(B) \in \mathscr{D}(P)\big\}\,, \qquad Q(B) = P\big(f^{-1}(B)\big)\,.$$

It is clear that Q is a probability measure on T. This Q is called the **image probability measure** induced by the f of P, and it is denoted by Pf^{-1} or fP. If P is complete, then $Q = Pf^{-1}$ is also complete.

Lemma 2.3 Suppose S is a complete separable metric space, T is a separable metric space, P is a regular probability measure on S, and f is a P-measurable mapping from S into T. Then the image mapping $Q = Pf^{-1}$ is a regular probability measure on T.

Proof It is evident that Q is a complete probability measure on T, and that $\mathscr{D}(Q)$ contains $\mathscr{B}(S)$. Accordingly, to show that Q is regular, one just needs to prove that "With respect to $B \in \mathscr{D}(Q)$ and $\epsilon > 0$, it is possible to take an appropriate compact set $K \subset B$ such that $Q(B - K) < \epsilon$." Let $B \in \mathscr{D}(Q)$. As $A = f^{-1}(B)$ is P-measurable, from the previous lemma, it is possible to take an appropriate compact set $H \subset A$ such that

$$P(A - H) < \epsilon\,, \qquad f_H = f|_H : H \longrightarrow T \text{ is continuous.}$$

Accordingly, $K = f_H(H) = f(H)$ is compact. Clearly, since $f^{-1}(K) \supset H$,

$$Q(B - K) = P\big(f^{-}(B - K)\big) = P\big(f^{-1}(B) - f^{-1}(K)\big) \leqq P(A - H) < \epsilon\,. \qquad ∎$$

Based on these preparations, we will prove the following theorem, which is our objective.

Theorem 2.6 The regular probability measure P on a complete separable metric space S is standard.

Proof The sets used in Lemma 2.1 are

$$B_{nk}, \qquad n, k = 1, 2, \cdots .$$

Arrange them as $B_1, B_2, \cdots$. Denote the indicator function of B_n as e_n. Now, define the real-valued function on S, $f : S \to \boldsymbol{R}^1$, as follows:

$$f(x) = \sum_{n=1}^{\infty} \frac{2e_n(x)}{3^n}.$$

$f(S)$ is a subset of the Cantor set $\boldsymbol{K}$. It is easy to prove–with the foundational knowledge of measure theory–that f is measurable, and we take it for granted. Let $Q = Pf^{-1}$. Since $\boldsymbol{R}^1$ is a separable metric space, from the previous lemma, Q is a regular probability measure.

What is left to prove is that (S, P) and $(\boldsymbol{R}^1, Q)$ are isomorphic. Let x and y be two different points of S. Since for any n, we have $x \in B_n$ and $y \notin B_n$, we have $e_n(x) \neq e_n(y)$. Accordingly, $f(x) \neq f(y)$. Hence f is injective. Restrict the range to $T = f(S) \left(\subset \boldsymbol{R}^1 \right)$ and once $f'S :\to T$ is established, f is bijective. Since $f^{-1}(T) = S$, $Q(T) = 1$, and $Q' = Q|_T$ is a probability measure on T. As $f(A) = f'(A) \subset T$, it can be immediately understood that

$$A \in \mathcal{D}(P) \Longleftrightarrow f'(A) \in \mathcal{D}(Q') \implies Q'\big(f'(A)\big) = P(A).$$

Consequently, we obtain

$$(S, P) \simeq (T, Q') \qquad (f'),$$

and eventually (S, P) and $(\boldsymbol{R}^1, Q)$ are isomorphic. ∎

Countable set Ω becomes complete and separable by the metric

$$\rho(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

In this case, $\mathcal{B}(\Omega) = 2^{\Omega}$, and it is alright to think that Theorem 2.5 is nested in Theorem 2.6.

The topology (product topology) of the product space $S = \prod_n S_r = n$ of the countable metric space $S_n = (S_n, \rho_n)$ (ρ_n is a metric, $n = 1, 2, \cdots$.) is given by

$$\rho\big((x_n), (y_n)\big) = \sum_{n=1}^{\infty} 2^{-n} [\rho_n(x_n, y_n) \wedge 1], \qquad a \wedge b = \min(a, b).$$

If $S_n = (S_n, \rho_n)$ is complete and separable, then $S = (S, \rho)$ is also complete and separable. Hence, it is understood that

$$\boldsymbol{R}^{\infty} = \boldsymbol{R}^1 \times \boldsymbol{R}^1 \times \cdots$$

is also a complete separable metric space. Since $\boldsymbol{R}^\infty$ is the space of all real-valued number sequences, sometimes it is called **sequence space**. From the above theorem, the regular probability measure on the sequence space is also standard.

The Borel set family $\mathcal{B}(\boldsymbol{R}^\infty)$ of $\boldsymbol{R}^\infty$ is denoted by $\mathcal{B}^\infty$. Now, let us prove that

$$\mathcal{B}^\infty = \mathcal{B}^1 \times \mathcal{B}^1 \times \cdots .$$

Since the n-mapping $\pi_n(x_1, x_2, \cdots) = x_n$ is continuous, with respect to all n, and all $E \in \mathcal{B}^1$, $\pi_n^{-1}(E)$ belongs to $\mathcal{B}^\infty$. It follows that

$$\mathcal{B}^\infty \supset \mathcal{B}^1 \times \mathcal{B}^1 \times \cdots .$$

To show that reverse inclusion relationship, note that the open set of $\boldsymbol{R}^\infty$ (with respect to product topology) can be expressed in the following form:

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times \boldsymbol{R}^1 \times \boldsymbol{R}^1 \times \cdots ,$$

which is a countable sum[1] of sets. This is of the same case as the countable product of separable metric spaces (in general, topological space that possesses countable open base)

$$(\Omega_n, P_n) \approx (\Omega_n', P_n') \quad (f_n), \quad n = 1, 2, \cdots .$$

If that is the case,

$$\left(\prod_n \Omega_n \prod_n P_n\right) \approx \left(\prod_n \Omega_n' \prod_n P_n'\right) \quad (f).$$

Here, f is defined as

$$f(\omega_1, \omega_2, \cdots) = \big(f_1(\omega_1), f_2(\omega_2), \cdots\big).$$

Moreover, let $\Omega = \prod_n \Omega_n$, $P = \prod_n P_n$ and $P_n(A_n) = 1$. Then, we have

$$\begin{aligned}
P\left(\prod_n A_n\right) &= P\Big(\lim_{n\to\infty} A_1 \times A_2 \times \cdots \times A_n \times \Omega_{n+1} \times \Omega_{n+2} \times \cdots\Big) \\
&= \lim_{n\to\infty} P\big(A_1 \times A_2 \times \cdots \times A_n \times \Omega_{n+1} \times \Omega_{n+2} \times \cdots\big) \\
&= \lim_{n\to\infty} P_1(A_1) \times P_2(A_2) \times \cdots \times P_n(A_n) = 1.
\end{aligned}$$

Applying the two results above, it is easy to prove that

$$(\Omega_n, P_n) \sim (\Omega_n', P_n') \implies \left(\prod_n \Omega_n \prod_n P_n\right) \sim \left(\prod_n \Omega_n' \prod_n P_n'\right)$$

Also, for the most part, the following is evident:

$$(\Omega, P) \sim (\Omega', P') \implies (\Omega, \bar{P}) \sim (\Omega', \bar{P}'), \qquad \big(\,\bar{}\, \text{ is Lebesgue extension}\big).$$

[1] product?

Now, let μ_n be Borel's probability measure on $\boldsymbol{R}^1$. The domain of product measure $\prod_{n=1}^{\infty} \mu_n$ is $\mathcal{B}^1 \times \mathcal{B}^1 \times \cdots$. As noted above, since this is equivalent to $\mathcal{B}^\infty = \mathbb{B}(\boldsymbol{R}^\infty)$, it follows that μ is the Borel probability measure on $\boldsymbol{R}^\infty$.

By Exercise 2.3 (ii)

$$\overline{\prod_{n=1}^{\infty} \mu_n} = \overline{\prod_{n=1}^{\infty}} \overline{\mu}_n.$$

From all these facts, the following theorems have been proved.

Theorem 2.7 Let P_n be the regular probability measure on the complete separable metric space S_n. Then the complete direct product of P_n $(n = 1, 2, \cdots)$, $P = \overline{\prod_n P_n}$ is a regular probability measure (accordingly a standard measure) on $S = \prod_n S_n$ (we have covered the fact that S is a complete separable metric space).

Theorem 2.8 The complete direct product $P = \overline{\prod_n P_n}$ of a standard probability measures P_n, $(n = 1, 2, \cdots)$ is also a standard probability measure.

Example 2.4 Let T be a Borel subset of the complete separable metric space S. The metric ρ in S that is restricted to T is denoted by ρ_T. With respect to ρ_T, T is a separable metric space. Show that the regular probability space (T, P) on T is standard.

[Hint] Let $Q(B) = P(B \cap T)$. From the regularity of $T \in \mathcal{B}(S)$ and P, Q is also a a regular probability measure on S. Accordingly, (S, Q) is standard. With $T \in \mathcal{B}(S)$, we obtain $T\ in \mathcal{D}(Q)$ and $Q(T) = 1$. It follows that P and $Q|_T$ are the same one, and we know that $(T, P) \sim (S, Q)$.

Christopher Ting
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