

2.3 Direct Product of Probability Measures

In the first chapter, the direct product of probability measures on finite sets is expressed in connection with the direct union of trials or independent probability variables. The same thing applies to generic trials.

First, one starts with the definition of the direct product of two probability measures. Suppose P_1 and P_2 are, respectively, the probability measures on Ω_1 and Ω_2 . Then, let

$$\Omega = \Omega_1 \times \Omega_2 .$$

The probability measure P on Ω , which has the product $\mathcal{D}(P_1) \times \mathcal{D}(P_2)$ of σ additive families as its domain, and satisfies

$$P(B_1 \times B_2) = P_1(B_1) P_2(B_2), \quad B_i \in \mathcal{D}(P_i), \quad i = 1, 2$$

is called the **direct product** of P_1 and P_2 , and denoted by $P_1 \times P_2$. The probability space (Ω, P) is called the direct product of (Ω_1, P_1) and (Ω_2, P_2) , and expressed as

$$(\Omega, P) = (\Omega_1, P_1) \times (\Omega_2, P_2) .$$

For example, the Lebesgue measure on the square $[0, 1]^2$ is the direct product of the Lebesgue measure on $[0, 1]$ with itself. In Chapter 1, $P = P_1 \times P_2$ is defined as

$$P\{(\omega_1, \omega_2)\} = P_1\{\omega_1\} P_2\{\omega_2\} .$$

From the above example of Lebesgues measure, one sees clearly that this definition is lacking.

The existence of the direct product and its uniqueness may be problematic. The totality \mathcal{I} of the subsets of Ω that take the form

$$B_1 \times B_2, \quad B_i \in \mathcal{D}(P_i), \quad i = 1, 2$$

is a multiplicative family. Moreover, it generates $\mathcal{B} \equiv \mathcal{D}(P_1) \times \mathcal{D}(P_2)$, and from the correspondence theorem concerning the probability measure, the uniqueness of the direct product is evident. To prove its existence, use the extension theorem in the previous section. First, with

$$P(B_1 \times B_2) = P_1(B_1) P_2(B_2),$$

define the set function p on \mathcal{I} , and then extend it unto the elementary probability measure on

$$\mathcal{A} = \left\{ A = \sum_{i=1}^n I_i \mid I_i \in \mathcal{I}, n = 1, 2, \dots \right\} \quad (\text{This is an additive family on } \Omega.)$$

If one can show that p possesses commonality, then p is extended unto the probability measure on $\sigma[\mathcal{A}] (= \mathcal{D}(P_1) \times \mathcal{D}(P_2))$, and this extension is the probability measure sought after.

The element A of \mathcal{A} is of course having the form

$$A = \sum_{i=1}^n B_{1,i} \times B_{2,i}, \quad B_{1,i} \in \mathcal{D}(P_1), \quad B_{2,i} \in \mathcal{D}(P_2), \quad i = 1, 2, \dots, n.$$

Moreover, $B_{1,i} (i = 1, 2, \dots, n)$ are mutually disjoint, and its sum can become Ω_1 . Here, it is possible that some $B_{2,i}$ are empty (refer to Figure 2.1). In this instance, if

$$p(A) \geq \alpha > 0,$$

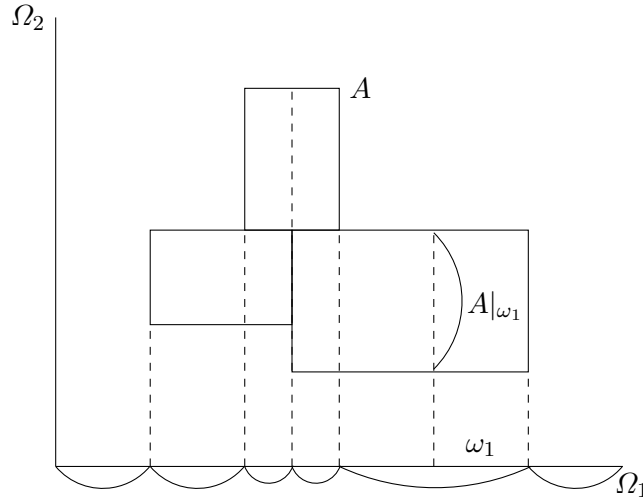


Figure 2.1

then the probability measure P_1 of

$$A_1(\alpha) = \sum B_{1,i}, \quad P_2(B_{2,i}) \geq \alpha/2$$

is above $\alpha/2$. The meaning of the above \sum is to sum over those i for which $P_2(B_{2,i}) \geq \alpha/2$ is satisfied. By rearranging the indices, let the totality of i be $1, 2, \dots, r$, then

$$\begin{aligned} \alpha &\leq p(A) = \sum_{i=1}^r P_1(B_{1,i}) P_2(B_{2,i}) + \sum_{i=r+1}^n P_1(B_{1,i}) P_2(B_{2,i}) \\ &\leq \sum_{i=1}^r P_1(B_{1,i}) \cdot 1 + \sum_{i=r+1}^n P_1(B_{1,i}) \cdot \alpha/2 \\ &\leq P_1(A_1(\alpha)) + P_2(A_1(\alpha)^c) \cdot \alpha/2 \\ &\leq P_1(A_1(\alpha)) + \alpha/2, \end{aligned}$$

and hence

$$P_1(A_1(\alpha)) \geq \alpha/2.$$

With this at the back of the mind, prove the commonality of p .

Now suppose the decreasing sequence of \mathcal{A} ,

$$A' \supset A'' \supset \dots \supset A^{(n)} \supset \dots$$

satisfies $p(A^{(n)}) \geq \alpha > 0$. Then it is evident that

$$A'_1(\alpha) \supset A''_1(\alpha) \supset A'''_1(\alpha) \supset \dots$$

From the note above, the P_1 measure of these sets is above $\alpha/2$. Accordingly,

$$P_1 \left(\bigcap_{n=1}^{\infty} A_1(\alpha) \right) = \lim_{n \rightarrow \infty} P_1(A_1^{(n)}(\alpha)) \geq \frac{\alpha}{2}.$$

Hence this intersection is not empty. Take a point ω_1 from it, and the fact that the measure P_2 of

$$A^{(n)}|_{\omega_1} = \{\omega_2 \in \Omega_2 | (\omega_1, \omega_2) \in A^{(n)}\} \quad (\text{the cut by } \omega_1 \text{ of } A^{(n)})$$

is above $\alpha/2$, is evident from the definition of $A^{(n)}(\alpha)$. Moreover, from the decreasing property of $\{A^{(n)}\}$, $\{A^{(n)}|_{\omega_1}\}$ is also decreasing.. Accordingly, like the previous cases, such set sequences have common points. If one of these is ω_2 , then (ω_1, ω_2) is a common point of $\{A^{(n)}\}$. Thus, the commonality of p is shown.

It is possible to define the direct product of three or more probability measures:

$$\begin{aligned} P_1 \times P_2 \times P_3 &= (P_1 \times P_2) \times P_3, \\ P_1 \times P_2 \times P_3 \times P_4 &= (P_1 \times P_2 \times P_3) \times P_4, \\ &\dots \end{aligned}$$

Obviously, $P_1 \times P_2 \times \dots \times P_n$ is determined by

$$\begin{aligned} \mathcal{D}(P_1 \times P_2 \times \dots \times P_n) &= \mathcal{D}(P_1) \times \mathcal{D}(P_2) \times \dots \times \mathcal{D}(P_n), \\ (P_1 \times P_2 \times \dots \times P_n)(B_1 \times B_2 \times \dots \times B_n) &= P_1(B_1) P_2(B_2) \dots P_n(B_n), \quad B_i \in \mathcal{D}(P_i), \quad i = 1, 2, \dots, n. \end{aligned}$$

At times $P_1 \times P_2 \times \dots \times P_n$ is written as $\prod_{i=1}^n P_i$. The direct product of a finite number of probability spaces can also be defined as

$$\prod_{i=1}^n (\Omega_i, P_i) = \left(\prod_{i=1}^n \Omega_i, \prod_{i=1}^n P_i \right) \quad \left(\prod_{i=1}^n \Omega_i = \Omega_1 \times \Omega_2 \times \dots \times \Omega_n \right)$$

Next, consider the direct product of countable probability measures. Suppose P_1, P_2, \dots are, respectively, the probability measures on $\Omega_1, \Omega_2, \dots$, and let

$$\Omega = \prod_{n=1}^{\infty} \Omega_n = \Omega_1 \times \Omega_2 \times \dots \times \Omega_n \times \dots$$

Let $\pi_n : \Omega \rightarrow \Omega_n$ be the projection. The σ additive family generated by the set family on Ω :

$$\pi_n^{-1}(B_n), \quad B_n \in \mathcal{D}(P_n), \quad n = 1, 2, \dots$$

is denoted by

$$\prod_{n=1}^{\infty} \mathcal{D}(P_n) \quad \text{or} \quad \mathcal{D}(P_1) \times \mathcal{D}(P_2) \times \dots$$

and is called the **product σ additive family** of $\mathcal{D}(P_n)$ ($n = 1, 2, \dots$). This definition of product additive family is applicable on any sequence of σ additive families. Now the domain of the probability measure P on Ω is $\prod_{n=1}^{\infty} \mathcal{D}(P_n)$. Moreover, when

$$P \left(\bigcap_{i=1}^n \pi_i^{-1}(B_i) \right) = \prod_{i=1}^n P_i(B_i), \quad B_i \in \mathcal{D}(P_i), \quad i = 1, 2, \dots, n, \quad n = 1, 2, \dots$$

holds, P is said to be the **product probability measure** of P_n ($n = 1, 2, \dots$), and is denoted by

$$P = \prod_{n=1}^{\infty} P_n \quad \text{or} \quad P_1 \times P_2 \times \dots$$

Let the totality of the subsets of Ω that can be written as

$$\bigcap_{i=1}^n \pi_i^{-1}(B_i) \quad \text{i.e.} \quad B_1 \times B_2 \times B_n \times \Omega_{n+1} \times \Omega_{n+2} \times \dots$$

be denoted by \mathcal{S} . The value of the direct product P on \mathcal{S} is given according to its definition. Moreover, \mathcal{S} is a multiplicative family, and since $\sigma(\mathcal{S})$ corresponds to the domain $\prod_{n=1}^{\infty} \mathcal{D}(P_n)$ of P , from the correspondence theorem of probability measure, even when P exists, it is the only one.

Now prove the existence of P by using the extension theorem. The totality of sets expressible as direct sums of the elements of \mathcal{S} is denoted by \mathcal{A} , which is an additive family, and generates $\mathcal{D}(P) \equiv \prod_{n=1}^{\infty} \mathcal{D}(P_n)$. With respect to an element of \mathcal{S} , it is a matter of fact to define

$$P(B_1 \times B_2 \times B_n \times \Omega_{n+1} \times \Omega_{n+2} \times \dots) = P_1(B_1)P_2(B_2) \cdots P_n(B_n).$$

Here, a remark is in order. Of course one should define

$$P(B_1 \times B_2 \times B_n \times \Omega_{n+1} \times \Omega_{n+2} \times \dots) = P_1(B_1)P_2(B_2) \cdots P_n(B_n)P_{n+1}(\Omega_{n+1})P_{n+2}(\Omega_{n+2}) \cdots$$

instead, nonetheless since $P_{n+1}(\Omega_{n+1})$ and so on are 1, both definitions are the same. Here, that P_n is a probability measure is critical. The P defined on \mathcal{S} can be extended unto an elementary probability measure on \mathcal{A} via a natural method. What remains is to further extend it to a probability measure on $\sigma[\mathcal{A}]$, i.e., $\prod_{n=1}^{\infty} \mathcal{D}(P_n)$. Consequently, one just need to prove that P possesses the property of commonality.

Let $A^{(1)}, A^{(2)}, \dots$ be the decreasing sequence of \mathcal{A} , and suppose $P(A^{(n)}) > \alpha > 0$ ($n = 1, 2, \dots$). Then if the existence of a point $\omega = (\omega_1, \omega_2, \dots) \in \Omega$ belonging to $A^{(n)}$ is proven, then P has a the property

of commonality. Similar to how the additive family \mathcal{A} on $\Omega_1 \times \Omega_2 \times \dots$ and the elementary probability measure are determined, suppose \mathcal{A}' and P' are determined on $\Omega_1 \times \Omega_2 \times \dots$. Then for any $A \in \mathcal{A}$, one can write

$$A = \sum_{i=1}^n B_{1,i} \times B_{2,i}, \quad B_{1,i} \in \mathcal{D}(P_1), \quad B_{2,i} \in \mathcal{A}', \quad \Omega_1 = \sum_{i=1}^n B_{1,i}.$$

and note that

$$P(A) = \sum_{i=1}^n P_1(B_{1,i})P'(B_{2,i}).$$

Using the same argument when the direct product of the two probability measures is determined, $\omega_1 \in \Omega$ is determined and

$$P'(A^{(n)}|_{\omega_1}) \geq \alpha/2, \quad n = 1, 2, \dots,$$

Accordingly one can have

$$A^{(n)}|_{\omega_1} \neq \emptyset, \quad n = 1, 2, \dots,$$

Here, $A^{(n)}|_{\omega_1}$ is the **section** by ω_1 of $A^{(n)}$, and it determines

$$A^{(n)}|_{\omega_1} = \{(\omega_2, \omega_3, \dots) | (\omega_1, \omega_2, \omega_3, \dots) \in A^{(n)}\} \in \mathcal{A}'.$$

$A^{(n)}|_{\omega_1}$ ($n = 1, 2, \dots$) is a decreasing sequence in $\Omega_2 \times \Omega_3 \times \dots$, and its P' measure is all above $\alpha/2$. By the same argument applied on $A^{(n)}|_{\omega_1}$, fix ω_2 to obtain

$$P''(A^{(n)}|_{(\omega_1, \omega_2)}) \geq \alpha/4.$$

Accordingly, one has

$$A^{(n)}|_{(\omega_1, \omega_2)} \neq \emptyset, \quad n = 1, 2, \dots$$

Here, P'' is an elementary probability measure on $\Omega'' = \Omega_1 \times \Omega_2 \times \dots$, and defined in the same way as P and P' . $A^{(n)}|_{(\omega_1, \omega_2)}$ is the section by (ω_1, ω_2) of $A^{(n)}$, and is defined by

$$A^{(n)}|_{(\omega_1, \omega_2)} = \{(\omega_3, \omega_4, \dots) | (\omega_1, \omega_2, \omega_3, \omega_4, \dots) \in A^{(n)}\}.$$

Repeating this procedure, fix ω_k ($k = 1, 2, \dots$), then

$$A^{(n)}|_{(\omega_1, \omega_2, \dots, \omega_k)} \neq \emptyset, \quad n = 1, 2, \dots, k = 1, 2, \dots$$

Let $\omega = (\omega_1, \omega_2, \dots)$, and prove that

$$\omega \in A^{(n)}, \quad n = 1, 2, \dots$$

As $A^{(n)} \in \mathcal{A}$, $A^{(n)}$ is a finite direct sum of the elements of \mathcal{G} . Accordingly, $A^{(n)}$ is of the form of

$$A^{(n)} = B \times \Omega_{N+1} \times \Omega_{N+2} \times \dots, \quad B \subset \Omega_1 \times \Omega_2 \times \dots \times \Omega_N.$$

Of course, N is related to n . From the way $\omega_1, \omega_2, \dots$ are fixed,

$$A^{(n)}|_{(\omega_1, \omega_2, \dots, \omega_N)} \neq \emptyset,$$

that is

$$(\omega_1, \omega_2, \dots, \omega_N) \in B.$$

Accordingly, for any $\omega'_{N+1}, \omega'_{N+2}, \dots$, one obtains

$$(\omega_1, \omega_2, \dots, \omega_N, \omega'_{N+1}, \omega''_{N+2}) \in B \times \Omega_{N+1} \times \Omega_{N+2} \times \dots = A^{(n)},$$

and in particular

$$\omega = (\omega_1, \omega_2, \dots, \omega_N, \omega_{N+1}, \omega_{N+2}, \dots) \in A^{(n)}.$$

Thus, it is shown that the elementary probability measure P on \mathcal{A} has the property of commonality.

Accordingly, the existence of the direct product $\prod_{n=1}^{\infty} P_n$ is also proven.

The direct product of measures can be consider for a generic measure. Suppose m is a measure on Ω . When $m(\Omega) < \infty$, it is called a **finite measure**. When $\{A_n\}$ exists such that

$$\Omega = \sum_{n=1}^{\infty} A_n, \quad m(A_n) < \infty, \quad n = 1, 2, \dots$$

it is called **σ finite measure**. When m is a finite measure, $P(A) = m(A)/m(\Omega)$ (when $m(\Omega) > 0$) is a probability measure, the direct product of finite measures reduces to the case of direct product of of probability measures. That is, suppose

$$m_1 \times m_2 = m_1(\Omega_1)m_2(\Omega_2) \frac{m_1}{m_1(\Omega_1)} \frac{m_2}{m_2(\Omega_2)}$$

(When $m_1(\Omega_1) = 0$ or $m_2(\Omega_2)$, $m_1 \times m_2$ is constantly equal to 0.), then

$$(m_1 \times m_2)(A_1 \times A_2) = m_1(A_1) m_2(A_2),$$

and $m_1 \times m_2$ has the property of the direct product of m_1 and m_2 . The uniqueness of such $m_1 \times m_2$ comes from the theorem of correspondence. By this reasoning, one can define the direct product of a finite number of finite measures, and when the number is infinite, in order to define

$$\prod_{n=1}^{\infty} m_n = \prod_{n=1}^{\infty} m_n(\Omega_n) \prod_{n=1}^{\infty} \frac{m_n}{m(\Omega_n)},$$

it is necessary for $\prod_{n=1}^{\infty} m_n(\Omega_n)$ to have meaning. To make the infinite product absolutely convergent, it is necessary and sufficient to have

$$\sum_{n=1}^{\infty} |m_n(\Omega_n) - 1| < \infty.$$

If m_1 and m_2 are respectively the σ finite measures on Ω_1 and Ω_2 , then $\{A_{in}\}$ exists such that

$$\Omega_i = \sum_{n=1}^{\infty} A_{in}, \quad m(A_{in}) < \infty, \quad n = 1, 2, \dots, \quad i = 1, 2.$$

holds. Accordingly

$$m_{in}(B_i) = m_{in}(B_1 \cap A_{in}), \quad i = 1, 2$$

is a finite measure on Ω_i , and $m_{1n} \times m_{2k}$ is determined.

If $m_1 \times m_2$ is defined as

$$(m_1 \times m_2)(A) = \sum_{n,k=1}^{\infty} (m_{1n} \times m_{2k})(A),$$

then $m_1 \times m_2$ has the property of a direct measure:

$$(m_1 \times m_2)(A_1 \times A_2) = m_1(A_1) m_2(A_2).$$

In the same way, $m_1 \times m_2 \times \dots \times m_n$ can be defined.

When trial sequence \mathbf{T}_n ($n = 1, 2, \dots$) (finite or countably infinite) is given, the sample space Ω of their direct union $\mathbf{T} = \mathbf{T}_1 \times \mathbf{T}_2 \times \dots$ is a direct product of the component sample spaces $\Omega_1, \Omega_2, \dots$. This is the same as the case of finite trials, and the probability law P of \mathbf{T} is given by

$$P = P_1 \times P_2 \times \dots, \quad P_n \text{ is the probability of } \mathbf{T}_n.$$

This is also the same as the case of finite trials, and should be recognized as an axiom. Accordingly, thinking about the direct union reduces to thinking about the direct product of the probability spaces of the component trials.

Exercise 2.3 (i) A dice is tossed continuously. Find the probability that on the n -th toss, the point 6 appears for the very time.

[Hint] First, determine the probability space (Ω, P) of this trial. Next, determine the subset A_n of Ω that describes the phenomenon in question, and then calculate $p_n = P(A_n)$. Even in this simple question, to solve it rigorously, it is noteworthy that the concept of the infinite product of probability measures is necessary.

(ii) When the Lebesgue extension of the probability measure P is written as \overline{P} , show that

$$\overline{P_1 \times P_2 \times \dots} = \overline{\overline{P_1} \overline{P_2} \times \dots}$$

is valid. ($\overline{P_1 \times P_2 \times \dots}$ is called the **complete direct product** of P_1, P_2, \dots)

(iii) Suppose \mathcal{B}_1 and \mathcal{B}_2 are, respectively, the σ additive families on Ω_1 and Ω_2 . Show that the section of $B \in \mathcal{B}_1 \times \mathcal{B}_2$ by $\omega_1 \in \Omega_1$ belongs to \mathcal{B}_2 .

[Hint] One just needs to show that

The totality of B possessing such property is a σ additive family that contains all of $B_1 \times B_2$ ($B_i \in \mathcal{B}_i, i = 1, 2$).

