§ 2.2 The Extension Theorem of Probability Space

It was understood in the previous section that the probability measure played an important role in the research of trials. As explained in the example of a trial of tossing a coin infinite number of times, for a certain simple set of the sample space, its probability can be relatively easy to define. The problem is to extend this probability to probability measure. This section's objective is to study this extension problem.

Suppose $\mathscr{A}$ is a certain subset of Ω. When $\mathscr{A}$ is closed with respect to intersection, i.e.,

$$A, B \in \mathscr{A} \Longrightarrow A \cap B \in \mathscr{A} .$$

Then $\mathscr{A}$ is called the **multiplicative family**. Furthermore, when $\mathscr{A}$ is closed with respect to countable direct union and eigen difference, and also contains Ω, then $\mathscr{A}$ is called the **Dynkin family**. Same as the σ additive family $\sigma[\mathscr{A}]$, which was defined as it being generated with respect to any family $\mathscr{A}$ of subsets, it is possible to define the Dynkin family $\delta[\mathscr{A}]$ as being generated by $\mathscr{A}$. Since a σ additive family is a Dynkin family, of course

$$\delta[\mathscr{A}] \supset \sigma[\mathscr{A}] .$$

Suppose $\mathscr{A}$ is a Dynkin family and a multiplicative family. Since $\mathscr{A}$ contains Ω,

$$A \in \mathscr{A} \Longrightarrow A^c = \Omega - A \in \mathscr{A} ,$$

and accordingly

$$A_n \in \mathscr{A} \ (n = 1, 2, \cdots, n) \Longrightarrow \bigcup_n A_n = \sum_n A_n \cap A_1^c \cap A_2^c \cap \cdots \cap A_{n-1}^c \in \mathscr{A} .$$

For this reason, $\mathscr{A}$ is a σ additive family.

Theorem 2.1 (Dynkin Theorem) If $\mathscr{A}$ is a multiplicative family, then $\delta[\mathscr{A}] = \sigma[\mathscr{A}]$.

Proof It is sufficient to prove $\delta[\mathscr{A}] \supset \sigma[\mathscr{A}]$. For this, one just needs to show that $\delta[\mathscr{A}]$ is a σ additive family, and to this end, one just needs to show that $\delta[\mathscr{A}]$ is a multiplicative family. Let

$$\mathscr{D}_1 = \left\{ A \middle| \text{for all } B \in \mathscr{A}, A \cap B \in \delta[\mathscr{A}] \right\} .$$

As $\delta[\mathscr{A}]$ is a Dynkin family, $\mathscr{D}_1$ is also a Dynkin family. Also, as $\mathscr{A}$ is a multiplicative family, $\mathscr{D}_1$ contains $\mathscr{A}$. Accordingly, from the definition of $\delta[\mathscr{A}]$, $\delta[\mathscr{A}] \subset \mathscr{D}_1$. This means that

$$A \in \delta[\mathscr{A}], B \in \mathscr{A} \Longrightarrow A \cap B \in \delta[\mathscr{A}] .$$

Accordingly, the set family

$$\mathscr{D}_2 = \left\{ B \middle| \text{for all } A \in \mathscr{A}, A \cap B \in \delta[\mathscr{A}] \right\}$$

contains $\mathscr{A}$. Because just like $\mathscr{D}_1$, $\mathscr{D}_2$ too is a Dynkin family, it follows that $\delta[\mathscr{A}] \subset \mathscr{D}_2$, and

$$A \in \delta[\mathscr{A}], B \in \delta[\mathscr{A}] \Longrightarrow A \cap B \in \delta[\mathscr{A}] ,$$

Namely, it is understood that $\delta[\mathscr{A}]$ is a multiplicative family. ▮

As an application of the Dynkin theorem, the following theorem is obtained.

Theorem 2.2 (**Correspondence Theorem Concerning Probability Measure**) Suppose P_1 and P_2 are the measures on Ω, and $\mathscr{A}$ is a multiplicative family, which is in $\mathscr{D}(P_1) \cap \mathscr{D}(P_2)$. If P_1 and P_2 are in correspondence on $\mathscr{A}$, then it is also in correspondence on $\sigma[\mathscr{A}]$. Namely,

$$P_1(A) = P_2(A)\,, \qquad A \in \mathscr{A} \Longrightarrow P_1(A) = P_2(A)\,, \qquad A \in \sigma[\mathscr{A}]\,.$$

Proof That the set family

$$\mathscr{B} = \big\{A \in \mathscr{D}(P_1) \cap \mathscr{D}(P_2) \big| P_1(A) = P_2(A)\big\}$$

is a Dynkin family is understood readily from the properties of probability measure. From the assumption $\mathscr{B} \supset \mathscr{A}$. Accordingly, $\mathscr{B} \supset \delta[\mathscr{A}]$. As $\mathscr{A}$ is a multiplicative family, from the previous theorem, $\delta[\mathscr{A}] = \sigma[\mathscr{A}]$. Hence $\mathscr{B} \supset \sigma[\mathscr{A}]$. This means that P_1 and P_2 are in correspondence on $\sigma[\mathscr{A}]$. ▮

The probability measure on a topological space such that its domain corresponds to the family of Borel sets is called the **Borel probability measure**, and the Lebesgue extension of the Borel probability measure is called the **regular probability measure**. Suppose P_1 and P_2 are the probability measures on the topological space S such that their respective domains $\mathscr{D}_1(P_1)$ and $\mathscr{D}_2(P_2)$ both contain $\mathscr{B}(S)$. Obviously the collection of open sets $\mathscr{G} = \mathscr{G}(S)$ is a multiplicative family, and $\mathscr{B}(S)$ is a σ additive family generated by $\mathscr{G}$. From the correspondence theorem, if P_1 and P_2 are in correspondence on $\mathscr{G}$, then they are in correspondence on $\mathscr{B}(S)$. Accordingly, if 2 Borel probability measures on S are in correspondence on $\mathscr{G}$, they are in perfect correspondence. Moreover, if P is a regular probability measure, the restriction $P|_{\mathscr{B}(S)}$ of P to $\mathscr{B}(S)$ (this is a Borel probability measure) is in correspondence to the Lebesgue extension. It follows also that if two regular probability measures on S are in correspondence, then they are in perfect correspondence. In other words

> "The Borel probability measure and the regular probability measure on S are completely determined by the actions on their family of open sets."

In the above discussion, instead of the family of open sets, if any multiplicative family (such as the family of closed sets) that generates $\mathscr{B}(S)$ is taken, then the result also holds.

Define $\mu^*(A)$ with respect to all the subsets A of Ω and when the following 4 conditions are met, the set function μ^* is called the **exterior measure**.

(C.1) $\qquad 0 \leqq \mu^*(A) \leqq \infty\,,$

(C.2) $\qquad \mu^*(\emptyset) = 0\,,$

(C.3) (**Monotonicity**) $\qquad A \subset B \Longrightarrow \mu^*(A) \leqq \mu^*(B)\,,$

(C.4) (**Inferior Additive Property**) $\qquad \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leqq \sum_{n=1}^{\infty} \mu^*(A_n).$

With respect to the exterior measure μ^*, A is said to be **measurable** (μ^*-measurable), when for any $W \subset \Omega$,

$$\mu^*(W) = \mu^*(W \cap A) + \mu^*(W \cap A^c) \,.$$

Under (C.4), let $A_{m+1} = A_{m+2} = \cdots = \emptyset$, from (C.2), since

$$\mu^*\bigl(A_1 \cup A_2 \cup \cdots \cup A_m\bigr) \leqq \mu^*(A_1) + \mu^*(A_2) + \cdots + \mu^*(A_m)$$

is obtained,

$$\mu^*(W) \leqq \mu^*(W \cap A) + \mu^*(W \cap A^c)$$

holds unconditionally. Accordingly, it is alright to rewrite the above inequality of measurability as

$$\mu^*(W) \geqq \mu^*(W \cap A) + \mu^*(W \cap A^c) \,.$$

Theorem 2.3 (Carathéodory's Theorem) Let μ^* be the exterior measure on Ω, and suppose $\mathscr{M}$ is the totality of all the sets that are μ^*-measurable. The μ^* restricted to $\mathscr{M}$, $\mu^*|_{\mathscr{M}}$, is a measure on Ω. In particular, when $\mu^(\Omega) = 1$, $\mu^*|_{\mathscr{M}}$ is a probability measure.

Proof Let $\mu = \mu^*|_{\mathscr{M}}$, and $\mathscr{D}(\mu) = \mathscr{M}$. First, show that $\mathscr{M}$ is a σ additive family. That $\mathscr{M}$ contains Ω, and that $\mathscr{M}$ is closed under set complement are easily understood. Let $A, B \in \mathscr{M}$,

$$\begin{aligned}
\mu^*(W) &\geqq \mu^*(W \cap A) + \mu^*(W \cap A^c) \\
&\geqq \mu^*(W \cap A) + \mu^*(W \cap A^c \cap B) + \mu^*(W \cap A^c \cap B^c) \\
&\mu^*\bigl(W \cap (A \cup B)\bigr) + \mu^*\bigl(W \cap (A \cup B)^c\bigr) \\
&\bigl(\because A \cup B = A + A^c \cap B\bigr)
\end{aligned}$$

This means that $A \cup B \in \mathscr{M}$. Accordingly, $\mathscr{M}$ is closed under finite union. If $A_1, A_2, \cdots, A_n \in \mathscr{M}$ are mutually disjoint, then, like above,

$$\begin{aligned}
\mu^*(W) &\geqq \mu^*(W \cap A_1) + \mu^*(W \cap A_1^c) \\
&\geqq \mu^*(W \cap A_1) + \mu^*(W \cap A_1^c \cap A_2) + \mu^*(W \cap A_1^c \cap A_2^c) \\
&\geqq \mu^*(W \cap A_1) + \mu^*(W \cap A_2) + \mu^*(W \cap A_1^c \cap A_2^c) \\
&\geqq \mu^*(W \cap A_1) + \mu^*(W \cap A_2) + \mu^*(W \cap A_1^c \cap A_2^c \cap A_3) + \mu^*(W \cap A_1^c \cap A_2^c \cap A_3^c) \\
&\geqq \mu^*(W \cap A_1) + \mu^*(W \cap A_2) + \mu^*(W \cap A_3) + \mu^*(W \cap A_1^c \cap A_2^c \cap A_3^c)
\end{aligned}$$

Repeating these, and

$$\begin{aligned}
\mu^*(W) &\geqq \sum_{n=1}^{N} \mu^*(W \cap A_n) + \mu^*(W \cap A_1^c \cap A_2^c \cap \cdots \cap A_n^c) \\
&\geqq \sum_{n=1}^{N} \mu^*(W \cap A_n) + \mu^*\left(W \cap \bigcap_{n=1}^{\infty} A_n^c\right) \qquad \text{(Monotonicity of } \mu^*\text{)}
\end{aligned}$$

Let $N \longrightarrow \infty$, then

$$\mu^*(W) \geqq \sum_{n=1}^{\infty} \mu^*(W \cap A_n) + \mu^* \left(W \cap \left(\sum_{n=1}^{\infty} A_n \right)^c \right)$$
$$\geqq \mu^* \left(W \cap \sum_{n=1}^{\infty} A_n \right) + \mu^* \left(W \cap \left(\sum_{n=1}^{\infty} A_n \right)^c \right) .$$

This means $\sum_{n=1}^{\infty} A_n \in \mathscr{M}$. Accordingly, $\mathscr{M}$ is closed under countable direct sum. From here, to derive that $\mathscr{M}$ is closed under countable sum, it is good to note

$$\bigcup_n A_n = \sum_{n=1}^{\infty} A_n \cap A_1^c \cap A_2^c \cap \cdots \cap A_n^c = \sum_{n=1}^{\infty} (A_n^c \cup A_1 \cup A_2 \cup \cdots \cup A_n)^c .$$

Thus, that $\mathscr{M}$ is σ additive family is proven.

Next is to show that $\mu = \mu^*|_{\mathscr{M}}$ fulfills the properties of measure. The only concern is the σ additivity. From the inequality obtained above, let

$$W = \sum_{n=1}^{\infty} A_n \qquad (A_1, A_2, \cdots \in \mathscr{M})$$

then

$$\mu \left(\sum_{n=1}^{\infty} A_n \right) \geqq \sum_{n=1}^{\infty} \mu(A_n) \geqq \mu \left(\sum_{n=1}^{\infty} A_n \right) .$$

As a result,

$$\mu \left(\sum_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n) .$$

Thus, it is understood that μ is a measure, and obvious that when $\mu^*(\Omega) = 1$, μ is a probability measure. ▮

Of the 3 conditions of the definition of σ additive family, relax the σ additivity (closed under finite sum) and define instead the **additive family**. When the function p defined on the additive family $\mathscr{A}$ on Ω satisfies

(p.1) $\quad p(A) \geqq 0$,

(p.2) (**additivity**) $\quad p(A + B) = p(A) + p(B)$,

(p.3) $\quad p(\Omega) = 1$,

it is called **elementary probability measure**. The probability measure on the finite set Ω in Chapter 1 is an elementary probability measure with 2^{Ω} as the domain. Since it is easy to derive the properties of an elementary probability measure, it is left to the readers. That σ additive family is an additive family, and that probability measure is an elementary measure are obvious from the definitions.

The totality of the finite direct sums of the sub-intervals of $[0, 1]$ is an additive family, and with respect to such sets. With respect to these sets, by treating the sum of lengths of the constituting intervals, an elementary probability measure is obtained. Moreover, the totality $\mathscr{A}$ of the finite direct sums of the sets in the set family $\mathscr{G}$ that resulted from introducing the trial of tossing a coin infinite number of times is an additive family too. With respect to

$$A = \sum_{i=1}^{n} I_i\,, \qquad I_i \in \mathscr{I}\,,$$

let

$$P(A) = \sum_{i=1}^{n} P(I_i)\,. \qquad\qquad (P(_i) \text{ was defined in the previous section.})$$

Then, P is an elementary probability measure on $\mathscr{A}$.

To define the probability measure as an extension of an elementary probability measure, the following theorem is needed.

Theorem 2.4 (**Extension Theorem of Probability Measure**) Suppose $\mathscr{A}$ is an additive family on a set Ω, and p is an elementary probability measure on $\mathscr{A}$. The necessary and sufficient condition for p to be extended to the probability measure P on $\sigma[\mathscr{A}]$ is that p is $\boldsymbol{\sigma}$ **additive** on $\mathscr{A}$. In other words

$$\text{“}A_1, A_2, \cdots \in \mathscr{A} \text{ are mutually disjoint, } \sum_{n=1}^{\infty} A_n \in \mathscr{A}\text{”} \implies p\left(\sum_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} p(A_n)$$

holds. The extension P is unique.

Note This condition is equivalent to

$$\text{“}A_1 \supset A_2 \supset \cdots, \supset A_n \in \mathscr{A}\ (n = 1, 2, \cdots,)\,, \bigcap_{n=1}^{\infty} A_n = \emptyset\text{”} \Longrightarrow \lim_{n \longrightarrow \infty} p(A_n) = 0\,.$$

Furthermore, this is equivalent to

$$\text{“}A_1 \supset A_2 \supset \cdots, \supset A_n \in \mathscr{A}\ (n = 1, 2, \cdots,)\,, \inf p(A_n) = \emptyset\text{”} \Longrightarrow \bigcap_{n=1}^{\infty} A_n \neq \emptyset\,.$$

This property is called the **commonality** of p.

Proof of Theorem 2.4 The extension's uniqueness comes from the theorem of correspondence concerning probability measures. Since it is clear that the above condition is necessary for extension to be possible, what remains is to prove the sufficiency. Let

$$\mu^*(B) = \inf\left\{ \sum_{n=1}^{\infty} p(A_n) \,\middle|\, \bigcup_{n=1}^{\infty} A_n \supset B, A_n \in \mathscr{A}\ (n = 1, 2, \cdots) \right\}.$$

It is easily understood that μ^* is an exterior measure, and that $\mu^*(\Omega)$. From Carathéodory's theorem, the totality $\mathscr{M}$ of μ^* measurable sets is a σ additive family, and $\mu = \mu^*|_{\mathscr{M}}$ is a probability measure. Accordingly, if one can say that

$$\mathscr{A} \subset \mathscr{M}\,, \qquad \mu^*(A) = p(A)\,, \quad A \in \mathscr{A}\,,$$

then $P = \mu^*|_{\mathscr{M}}$ is the sought after extension. (Note that from $\mathscr{A} \subset \mathscr{M}$, $\mathscr{D}(\mu) = \mathscr{M} \supset \sigma[\mathscr{A}]$ emerges.)

From the assumption, p is σ additive. Moreover, it is **inferior** $\boldsymbol{\sigma}$ **additive**. In other words, note that

$$
\begin{aligned}
&A \in \mathscr{A}\,, \quad A_n \in \mathscr{A}\,(n=1,2,\cdots)\,, \quad A \subset \bigcup_{n=1}^{\infty} A_n \\
&\quad \Longrightarrow A = \sum_{n=1}^{\infty} A \cap \left(A_n - \bigcup_{k=1}^{n-1} A_k \right) \\
&\quad \Longrightarrow p(A) = \sum_{n=1}^{\infty} p\left(A \cap \left(A_n - \bigcup_{k=1}^{n-1} A_k \right)\right) \\
&\quad \Longrightarrow p(A) \leqq \sum_{n=1}^{\infty} p(A_n)\,.
\end{aligned}
$$

From this, it is immediate that for all $A \in \mathscr{A}$, $p(A) \leqq \mu^*(A)$. Moreover, since

$$A = A \cup \emptyset \cup \emptyset \cup \cdots\,,$$

it emerges that

$$\mu^*(A) \leqq p(A) + p(\emptyset) + p(\emptyset) + \cdots = p(A)\,.$$

In the end, for all $A \in \mathscr{A}$, $\mu^*(A) = p(A)$.

For any W,

$$W \subset \bigcup_{n=1}^{\infty} A_n\,.$$

Take $A_n \in \mathscr{A}$ $(n=1,2,\cdots)$, and for any $A \in \mathscr{A}$,

$$
\begin{aligned}
&W \cap A \subset \bigcup_{n=}^{\infty} A_n \cap A\,, &&\text{hence,} \quad \mu^*(W \cap A) \leqq \sum_{n=1}^{\infty} p(A_n \cap A)\,, \\
&W \cap A^c \subset \bigcup_{n=}^{\infty} A_n \cap A^c\,, &&\text{hence,} \quad \mu^*(W \cap A^c) \leqq \sum_{n=1}^{\infty} p(A_n \cap A^c)\,.
\end{aligned}
$$

Adding both sides,

$$\mu^*(W \cap A) + \mu^*(W \cap A^c) \leqq \sum_{n=1}^{\infty} p(A_n)\,.$$

Taking the lower bound of the right hand side,

$$\mu^*(W \cap A) + \mu^*(W \cap A^c) \leqq \mu^*(W)\,.$$

This means that $A \in \mathscr{M}$. Accordingly, $\mathscr{A} \subset \mathscr{M}$, and thus the sufficient condition is proven. ▮

Applying the extension theorem above, prove the existence of Lebesgue-Stieltjes measure. Suppose P is a probability measure on $\boldsymbol{R}^1$, and its domain corresponds to $\mathscr{B}(\boldsymbol{R}^1)$. Such probability measure is called the **Borel probability measure** on $\boldsymbol{R}^1$, and the Lebesgue extension of the Borel probability measure is called the **regular probability measure**. It has already been mentioned that the Borel measure and the regular probability measure on a generic topological space could also be defined in the same way. $\mathscr{B}(\boldsymbol{R}^1)$ is usually abbreviated as $\mathscr{B}^1$, and similarly, $\mathscr{B}(\boldsymbol{R}^n)$ is written as $\mathscr{B}^n$.

Now when P is the regular probability measure on $\boldsymbol{R}^1$, the function

$$F(x) = F_P(x) = P\big((-\infty, x]\big)\,, \qquad x \in \boldsymbol{R}^1$$

is called the **distribution** of P. That $F(x)$ satisfies the following 3 conditions is easily understood from the properties of the probability measure.

(F.1) (Monotonicity) $\quad x \leqq y \implies F(x) \leqq F(y)$,

(F.2) (Right Continuity) $\quad F(x+) = F(x) \qquad \Big(F(x+) = \lim_{y \downarrow x} F(y)\Big)$

(F.3) $\quad F(-\infty) = 0\,,\quad F(\infty) = 1, \quad \Big(F(-\infty) = \lim_{x\downarrow -\infty} F(x)\,,\ F(\infty) = \lim_{x \uparrow \infty} F(x)\Big)$.

Conversely when $F(x)$ that satisfies these 3 conditions is given, there is only one regular probability measure P for which $F(x)$ is its distribution function. This P that corresponds to $F(x)$ is called the **Lebesgue-Stieltjes measure**. The existence of P is proven by the above extension theorem as follows. First, for the left half open interval $(a, b]$, define

$$p\big((a,b]\big) = F(b) - F(a)\,, \qquad -\infty \leqq a < b \leqq \infty\,.$$

Note that $(a, \infty]$ is taken as (a, ∞). Denote the totality of sets (for the time being name them as **elementary sets**) expressed as finite direct sums of left half open intervals by $\mathscr{A}$, which is an additive family. With respect to

$$A = \sum_{i=1}^{n} I_i\,, \qquad I_i \text{ is a left half open interval}\,,$$

define

$$p(A) = \sum_{i=1}^{n} p(I_i)\,.$$

There are many methods to decompose the element A of $\mathscr{A}$ like above. Correspondence to the decomposition, $p(A)$ found by the formula above is made to be unrelated to the method of decomposition. If the commonality of p holds, then p can be extended into a probability measure on $\mathscr{B}(S)$. Furthermore, if the Lebesgue extension is P, which is the probability measure one is after. Show that p possesses commonality. Let

$$A_1 \supset A_2 \supset \cdots\,, \qquad A_n \in \mathscr{A}\ (n = 1, 2, \cdots)\,, \qquad \alpha = \inf p(A_n) > 0\,.$$

Since F is right continuous, for any finite left half open interval $I = (a, b]$, with $J = (a + \epsilon, b]$,

$$\overline{J} \subset I\,, \quad 0 < p(I) - p(J) = F(a+\epsilon) - F(a) \qquad \big(\overline{J} \text{ is the closure of } J\big)\,.$$

By making ϵ smaller, $p(I) - p(J)$ can be made as small as possible as F is right continuous. Though it is the same when I is an infinite interval, by Condition (F.3), J can be taken as a finite interval. Since

$A_n \in \epsilon\mathscr{A}$ is a finite sum of left half open intervals, for each of the constituting interval, take the above J, and take their union (direct sum) B_n, resulting in

$$\overline{B}_n \subset A_n\,.$$

Moreover since $p(A_n) - p(B_n)$ can be made as small as possible, one fixes $\{B_n\}$ in such a way that

$$p(A_n) - p(B_n) < 2^{-n-1}\alpha\,, \qquad n = 1, 2\cdots$$

$$p(A_n) - p\left(\bigcap_{i=1}^{n} B_i\right) = p\left(\bigcup_{i=1}^{n}(A_n - B_i)\right) \leqq p\left(\bigcup_{i=1}^{n}(A_i - B_i\right)$$
$$\leqq \sum_{i=1}^{n} p(A_i - B_i) < \sum_{i=1}^{n} 2^{-i-1}\alpha < \frac{\alpha}{2}\,.$$

Since $p(A_n) \geqq \alpha$,

$$p\left(\bigcap_{i=1}^{n} B_i\right) > \frac{\alpha}{2}\,,$$

Accordingly, of course

$$\bigcap_{i=1}^{n} B_i \neq \emptyset\,.$$

Consequently,

$$\bigcap_{i=1}^{n} \overline{B}_i \neq \emptyset\,, \qquad n = 1, 2, \cdots$$

Since $\overline{B_i}$ is a bounded closed set, from Cantor's commonality theorem,

$$\bigcap_{n=1}^{\infty} B_n \neq \emptyset\,.$$

As $A_n \supset \overline{B_n}$, $\bigcap\limits_{n=1}^{\infty} A_n \neq \emptyset$ is obtained, and the common point property of p is proven.

Hence, the existence of the extension P is known, its uniqueness is evident from the correspondence theorem about the probability measure. (Note that the totality of left half open intervals is a multiplicative family, and moreover generates the family of Borel sets.)

Exercise 2.2 Prove that the set function $P(I)$, $I \in \mathscr{I}$, which was introduced with respect to the trial of tossing a coin infinite number of times in the previous section, can be extended to a probability measure on Ω.

[Hint] The totality $\mathscr{A}$ of the sets, which are expressible as the finite direct sums of the elements of $\mathscr{I}$, is an additive family, and P extended onto $\mathscr{A}$ by the natural method is an elementary probability measure. Accordingly, one just need to say something about the commonality of this elementary probability measure. Now, suppose

$$\varphi : \Omega \longrightarrow \boldsymbol{K} \quad \text{(Cantor set)}\,, \qquad\qquad (\omega_1, \omega_2, \cdots) \mapsto \sum_{n=1}^{\infty} \frac{2\omega_n}{3^n}\,,$$

then φ is a 1 to 1 correspondence. For $A \in \mathscr{A}$, $\varphi(A)$ is a closed subset of $\boldsymbol{K}$, and since $\boldsymbol{K}$ is compact, the commonality of P results from Cantor's commonality theorem.

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Translation Version 1.0
Date: 10 May, 2017