Chapter 2 Probability Measure

§ 2.1 General Trial and Probability Measure

In Chapter 1, the fundamental ways of thinking in probability theory concerning finite trials were explained. Henceforth, the trials are not limited to finite cases. Though general trials are considered, the basic ways of thinking are completely the same.

Consider a generic trial $\boldsymbol{T}$, and suppose its sample space os Ω, and its probability law is P. There is no need to explain sample space, only that it may be an infinite set. The probability law is also the same as in the case of finite trial, being a function of the subsets of Ω. albeit different in a few aspects.

In the case of finite trial, with respect to all $A \subset \Omega$, the probability $P(A)$ of sample points in A, which are the results of a trial, is always defined. This aspect is however not necessary in general, and it is sufficient to define $P(A)$ for a certain type of sets. The totality of all sets $A \subset \Omega$ on which $P(A)$ can be defined is the domain of the set function P, and it is denoted by $\mathscr{D}(P)$, which is not meant for any arbitrary set families, but those assumed to satisfy the following conditions:

(σ.1) $\quad \Omega \in \mathscr{D}(P)$,

(σ.2) $\quad A \in \mathscr{D}(P) \Longrightarrow A^c \in \mathscr{D}(P)$,

(σ.3) $\quad A_n \in \mathscr{D}(P)\ (n = 1, 2, \cdots) \Longrightarrow \bigcup_{n=1}^{\infty} A_n \in \mathscr{D}(P)$.

The first condition is taken for granted. The second condition is a requirement to have the probability of complement event A^c also defined if the probability of A is defined, which is again taken for granted. The third is a requirement to have the probability definable for the event $\bigcup_n A_n$ of at least one of the events amongst $A_1, A_2, \cdots$, if probabilities can be respectively defined for all of $A_1, A_2, \cdots$. Again, this is a natural assumption. In general, the set family that satisfies the three conditions of (σ.1), (σ.1), and (σ.1) is called the **σ additive family**. It is the most basic concept in measure theory.

The following assumptions are made for the value of P. When written as $P(A)$, A is assumed to belong to $\mathscr{D}(P)$, and

(P.1) $\quad P \geqq 0\,,$

(P.2) (**σ additive property**) $\quad P\left(\sum_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)\,,$

(P.3) $\quad P(\Omega) = 1\,.$

The first and second conditions are the same as in the case of finite trial. The second condition requires that when the events $A_1, A_2, \cdots$ are mutually exclusive, the probability of at least one of these events occurs is equal to the sum of all the respective probabilities of these events. In the case of finite trial, when two events are mutually exclusive (hence a finite number of mutually events), the same thing is required; but here the requirement is on countably infinite number of mutually exclusive events. In fact, in the case of finite trial, since Ω is a finite set, if $A_1, A_2, \cdots$ are mutually exclusive, then except a finite number of these sets, of course the rest are empty sets. So even if σ additive property is required, it is not essentially different from the additive property. The set functions that satisfy (P.1) and (P.2) are called **measure**. Furthermore, those that also satisfy (P.3) are called **probability measure**. The sets that belong to $\mathscr{D}(P)$ are called ***P*-measurable sets**. $P(A)$ is called the the ***P* measure** (or **measure**. The space Ω endowed with the probability measure P is called the **probability space** (Ω, P).

Suppose a probability space (Ω, P) is given apart from trials. Now, from Ω, a point is drawn, and assume that the draw is such that the probability is equal to the probability of a point drawn from $A \in \mathscr{D}(P)$. It is regarded as a trail, and is called the **draw from** (Ω, P). The sample space of this trial is of course Ω, and the probability laws is P. When the sample space of the trial $\boldsymbol{T}$ is Ω, and the probability space is P, $\boldsymbol{T}$ is essentially no different from the draw from (Ω, P).

Suppose $\Omega = [0, 1]$ and P is "the Lebesgue measure on $[0, 1]$." P is obviously a measure on Ω, and $\mathscr{D}(P)$ is the totality of all the Lebesgue measurable sets. In this instance, the draw from (Ω, P) is a random draw of a point from $[0, 1]$. Here, the reason for taking P as the Lebesgue measure is because each point is being drawn in a uniform fashion. To attach weight to the point during the draw is to set

$$P(E) = \int_E f(\omega)\, d\omega \ .$$

Here, $f(\omega)$ is a Lebesgue measurable function that satisfies

$$f(\omega) \geqq 0\,, \qquad \int_{[0,1]} f(\omega)\, d\omega = 1\,.$$

It is called **probability density**. Such probability measure is called the probability measure of the density f on $[0, 1]$. The above Lebesgue measure corresponds to the case of $f(x) \equiv 1$.

Even when the measure is said to be on $\Omega = [0, 1]$, it need not necessarily be restricted to density. For example, let a be a fixed point of Ω, and let

$$P(E) = \delta_a(E) = \begin{cases} 1, & a \in E\,, \\ 0, & a \notin E\,. \end{cases}$$

This too is a probability measure on Ω. In this instance $\mathscr{D}(P) = 2^\Omega$ (=family of all the subsets of Ω). In the draw from (Ω, δ_a), the probability of drawing a is 1, and the probability of drawing $\Omega - \{a\}$ is 0. Accordingly, even though the draw is no longer random, it is nonetheless unmistakably a draw.

As shown above, there are many methods to draw from $[0, 1]$. When referring to a **random** draw, ordinarily one is alluding to the case for which the draw is in accordance with the initially discussed Lebesgue measure.

Having given an extreme draw of drawing just one point a, consider the trial of drawing rational numbers as a more generic example. Label the rational numbers in $\Omega = [0,1]$ as $r_1, r_2, \cdots$. Let P be

$$P(E) = \sum_{i: r_i \in E} 2^{-i} .$$

It is easy to prove that this is the probability measure on Ω (readers please attempt). Again in this case, $\mathscr{D}(P) = 2^{\Omega}$. Denote the totality of all the rational number is Ω as $\boldsymbol{Q}_{[0,1]}$. Since

$$P\big(\boldsymbol{Q}_{[0,1]}\big) = 1 , \qquad P\big(\Omega - \boldsymbol{Q}_{[0,1]}\big) = 0 ,$$

the draw from (Ω, P) is almost certainly that a rational number is drawn.

As another example of an infinite trial, consider the trial of "tossing a coin infinite number of times." As always, let the head be 1, and the tail be 0. Then the sample point of this trial is a infinite sequence of 0's and 1's. For example, the sample point

$$(0, 0, 1, 0, 1, \cdots)$$

means

tail, tail, head, tail, head, $\cdots$.

Accordingly, the sample space Ω is

$$\Omega = \big\{\omega = (\omega_1, \omega_2, \cdots) \big| \omega_n = 0, 1 \, (n = 1, 2, \cdots)\big\} .$$

The probability law P of interest is entered in a way that suits such as trial.

Whether head or tail, the probability is $1/2$ for the first toss. These events on Ω are, respectively, expressed as

$$\Omega_1 = \big\{\omega = (1, \omega_2, \omega_3, \cdots) \big| \omega_n = 0, 1 \, (n = 2, 3, \cdots)\big\} ,$$
$$\Omega_0 = \big\{\omega = (0, \omega_2, \omega_3, \cdots) \big| \omega_n = 0, 1 \, (n = 2, 3, \cdots)\big\} ,$$

Accordingly,

$$P(\Omega_0) = P(\Omega_1) = \frac{1}{2} .$$

By the same token, for the first and second tosses, the probabilities of head head, head tail, tail head, tail tail are all $1/4$. Accordingly, let

$$\Omega_{ij} = \{\omega = (i, j, \omega_3, \omega_4, \cdots) \big| \omega_n = 0, 1\, (n = 3, 4, \cdots)\}\,, \qquad i, j = 0, 1\,.$$

Then

$$P(\Omega_{00}) = P(\Omega_{01}) = P(\Omega_{10}) = P(\Omega_{11}) = \frac{1}{4}\,.$$

In general, with respect to

$$\Omega_{i_1 i_2, \cdots, i_k} = \{\omega = (i_1, i_2, \cdots, i_k, \omega_{k+1}, \omega_{i+2}, \cdots) \big| \omega_n = 0, 1\, (n = k, k+1, \cdots)\}\,, \quad i_1, i_2, \cdots, i_k = 0, 1\,,$$

$$P(\Omega_{i_1, i_2, \cdots, i_k}) = 2^{-k}\,.$$

The set family obtained in this way is

$$\{\Omega_{i_1 i_2, \cdots, i_k} \big| i_\nu = 0, 1 (\nu = 1, 2, \cdots, k), k = 1, 2, \cdots\}\,,$$

and denoted by $\mathscr{I}$. Thus, with respect to an element I of $\mathscr{I}$, $P(I)$ is defined.

If the probability measure of an extended P is obtainable, then such probability measure is the probability law sought after. For this purpose, since it is necessary to discuss the extension theorem in the next section, it shall not be discussed here (refer to Exercise 2.2). The following example is a revision of measure theory, which is often alluded to throughout this book.

Exercise 2.1 (i) Show that with respect to Lebesgue measurable subset E of the set $\boldsymbol{R}^1$ of real numbers, $N_{m,v}, C_{m,c}$ as defined in the following are both the probability measures on $\boldsymbol{R}^1$. Here $m \in \boldsymbol{R}^1$; $v > 0$ and $c > 0$ are parameters.

Gauss Distribution $\quad N_{m,v}(E) = \int_E \frac{1}{\sqrt{2\pi\, v}} e^{-(x-m)^2/(2v)}\, dx\,,$

Cauchy Distribution $\quad C_{m,c}(E) = \int_E \frac{c}{\pi} \frac{1}{c^2 + (x-m)^2}\, dx\,.$

(ii) Denote the σ additive family on the set S of $\mathscr{B}$. Show that $\mathscr{B}$ contains $\emptyset, S$, countable union, countable intersection, and it is closed with respect to various operations of se complement, set difference, sup, and inf. A certain operation is said to be **closed** on a certain set family when the set obtained from applying the operation on a set of the set family is also a set of the set family.

[Hint] The fact that $\mathscr{B}$ contains S, and it is closed with regard to the two operations of countably infinite union and complement is the very definition of σ additive family. That $\emptyset \in \mathscr{B}$ is because $\emptyset = S^c$. Accordingly when $A, B \in S$,

$$A \cup B = A \cup B \cup \emptyset \cup \emptyset \cup \cdots \in \mathscr{B}\,,$$

and $\mathscr{B}$ is also closed with respect to finite union. Note the following matters when one is to say something about the closure on $\mathscr{B}$ of other operations:

$$\bigcap_n A_n = \left(\bigcap_n A_n^c\right)^c, \qquad A \backslash B = A \cap B^c,$$
$$\limsup_{n\longrightarrow\infty} A_n = \bigcap_k \bigcup_{n>k} A_n, \qquad \liminf_{n\longrightarrow\infty} A_n = \bigcup_k \bigcap_{n>k} A_n.$$

(iii) Prove that the probability measure P possesses the following properties:

$P(\emptyset) = 0, \qquad P(A^c) = 1 - P(A), \qquad 0 \leqq P(A) \leqq 1,$

$P(A+B) = P(A) + P(B) \qquad \big($**finite additive property**$\big),$

$A \subset B \Longrightarrow P(A) \leqq P(B) \qquad \big($**monotonicity**$\big), \qquad P(B-A) = P(B) - P(A),$

If A_n is a monotonic sequence, then $P\Big(\lim_{n\longrightarrow\infty} A_n\Big) = \lim_{n\longrightarrow\infty} P(A_n) \qquad \big($**monotonic limit theorem**$\big),$

$P\Big(\liminf_{n\longrightarrow\infty} A_n\Big) \leqq \liminf_{n\longrightarrow\infty} P(A_n) \leqq \limsup_{n\longrightarrow\infty} P(A_n) \leqq P\Big(\limsup_{n\longrightarrow\infty} A_n\Big) \qquad \big($**Fatuo's lemma**$\big),$

If $A = \lim_{n\longrightarrow\infty} A_n$, then $P(A) = \lim_{n\longrightarrow\infty} P(A_n) \qquad \big($**continuity**$\big),$

$P\Big(\bigcup_n A_n\Big) \leqq \sum_n P(A_n) \qquad \big($**inferior additive property**$\big),$

$P\Big(\bigcap_n A_n\Big) \geqq 1 - \sum_n P(A_n^c),$

$P(A_n) = 0 \; (n = 1, 2, \cdots) \Longrightarrow P\Big(\bigcup_n A_n\Big) = 0,$

$P(A_n) = 1 \; (n = 1, 2, \cdots) \Longrightarrow P\Big(\bigcap_n A_n\Big) = 1.$

[Hint] Since $\Omega = \Omega + \emptyset + \emptyset + \cdots$, from the σ additive property of P,

$$1 = 1 + P(\emptyset) + P(\emptyset) + \cdots = \;.$$

Therefore $P(\emptyset) = 0$. Accordingly,

$$P(A+B) = P(A+B+\emptyset+\emptyset+\cdots) = P(A) + P(B) + P(\emptyset) + P(\emptyset) + \cdots = P(A) + P(B).$$

From here, the first 3 rows of properties come through. To demonstrate monotonic limit theorem, first note that if the sequence is monotonically increasing,

$$A_1 \subset A_2 \subset \cdots \Longrightarrow \lim_{n\longrightarrow} A_n = A_1 + (A_2 - A_1) + (A_3 - A_2) + \cdots.$$

If the sequence is monotonically decreasing, converted it to the monotonically increasing sequence by taking the set complement. Then, derive the Fatou's lemma and continuity. To prove inferior additive property, note that

$$\bigcup_n A_n = \sum_n B_n\,, \qquad \text{where} \qquad B_n = A_n - \bigcup_{i=1}^{n-1} A_i \subset A_n\,.$$

Then derive the remaining properties.

(iv) When P is a probability measure, prove the following inequalities:

$$P\Big(\bigcup_n A_n \Big\backslash \bigcup_n B_n\Big) \leqq \sum_n P(A_n \backslash B_n)\,,$$
$$P\Big(\bigcap_n A_n \Big\backslash \bigcap_n B_n\Big) \leqq \sum_n P(A_n \backslash B_n)\,.$$

[Hint] Let $B = \bigcup_n B_n$. Since $B \supset B_n \quad (n = 1, 2, \cdots)$,

$$\bigcup_n A_n \backslash B = \bigcup_n (A_n \backslash B) \subset \bigcup_n (A_n \backslash B_n)\,.$$

Derive the first equation from here. For the second equation, convert to the first equation by using $A \backslash B = A \cap B^c = B^c \backslash A^c$.

(v) prove the following equivalence relations:

$x \in \limsup A_n \iff x$ is contained in an infinite number of sets amongst $A_1, A_2, \cdots$

$x \in \liminf A_n \iff x$ is contained in all sets except a finite number of sets amongst $A_1, A_2, \cdots$

(vi) When $\mathscr{A}$ is a collection of any subsets of S, show that there exits a smallest σ additive family amongst all the families that contains $\mathscr{A}$ (This is called the **$\boldsymbol{\sigma}$ additive family generated by** $\mathscr{A}$, and is denoted by $\sigma[\mathscr{A}]$).

[Hint] Take the common components of all σ additive families containing $\mathscr{A}$.

(vii) (**Family of Borel Sets**) When S is a topological space, the σ additive family generated by the family of the open sets of S is called the **family of Borel sets** , and is denoted by $\mathscr{B}(S)$. An element of $\mathscr{B}(S)$ is called the Borel set of S. Show that any interval of $\boldsymbol{R}^1$ is a Borel set. Moreover, show that $\mathscr{B}(\boldsymbol{R}^1)$ is generated by a bounded open set.

(viii) (**Product $\boldsymbol{\sigma}$ Additive Family**) When $\mathscr{B}_1$, $\mathscr{B}_2$, $\cdots$, $\mathscr{B}_n$ are, respectively, the σ additive families on $S_1, S_2, \cdots, S_n$, the σ additive family on $S_1 \times S_2 \times \cdots S_n$ generated by the family of subsets of $S_1 \times S_2 \times \cdots \times S_n$

$$\big\{B_1 \times B_2 \times \cdots \times B_n \big| B_1 \in \mathscr{B}_1, B_2 \in \mathscr{B}_2, \cdots B_n \in \mathscr{B}_n, \big\}$$

is called the **Product σ additive family**, and it is expressed as

$$\mathscr{B}_1 \times \mathscr{B}_2 \times \cdots \times \mathscr{B}_n \,.$$

Prove that

$$\mathscr{B}\big(\boldsymbol{R}^{d_1+d_2+\cdots+d_n}\big) = \mathscr{B}\big(\boldsymbol{R}^{d_1}\big) \times \mathscr{B}\big(\boldsymbol{R}^{d_2}\big) \times \cdots \times \mathscr{B}\big(\boldsymbol{R}^{d_n}\big)\,.$$

[Hint] If $\mathscr{B}\big(\boldsymbol{R}^{d_1+d_2}\big) = \mathscr{B}\big(\boldsymbol{R}^{d_1}\big) \times \mathscr{B}\big(\boldsymbol{R}^{d_2}\big)$ can be proven, then the general case can be derived via induction with respect to n.

Since the totality of $E_1 \subset \boldsymbol{R}^{d_1}$ such that $E_1 \times \boldsymbol{R}^{d_2} \in \mathscr{B}\big(\boldsymbol{R}^{d_1+d_2}\big)$ is the σ additive family that contains the family of open sets of $\boldsymbol{R}^{d_1}$, it also contains $\mathscr{B}(\boldsymbol{R}^{d_1})$. This is expressed as

$$E_1 \in \mathscr{B}\big(\boldsymbol{R}^{d_1}\big) \Longrightarrow E_1 \times \boldsymbol{R}^{d_2} \in \mathscr{B}\big(\boldsymbol{R}^{d_1+d_2}\big)\,.$$

By the same token,

$$E_2 \in \mathscr{B}\big(\boldsymbol{R}^{d_2}\big) \Longrightarrow \boldsymbol{R}^{d_1} \times E_2 \in \mathscr{B}\big(\boldsymbol{R}^{d_1+d_2}\big)\,.$$

Accordingly for $i = 1, 2$,

$$E_i \in \mathscr{B}\big(\boldsymbol{R}^{d_i}\big) \Longrightarrow E_1 \times E_2 = \big(E_1 \times \boldsymbol{R}^{d_2}\big) \cap \big(\boldsymbol{R}^{d_1} \times E_2\big) \in \mathscr{B}\big(\boldsymbol{R}^{d_1+d_2}\big)\,.$$

Thus

$$\mathscr{B}\big(\boldsymbol{R}^{d_1}\big) \times \mathscr{B}\big(\boldsymbol{R}^{d_2}\big) \subset \mathscr{B}\big(\boldsymbol{R}^{d_1+d_2}\big)\,.$$

The reverse inclusion relation is from the following

" $\mathscr{B}\big(\boldsymbol{R}^{d}\big)$ is generated by all the sets that have the form $(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d)$."

(ix) (**Measurable Mapping**) Suppose f is a mapping from S_1 into S_2; $\mathscr{B}_1$ and $\mathscr{B}_2$ are, respectively, the σ additive families of S_1 and S_2. When

$$E \in \mathscr{B} \Longrightarrow f^{-1}(E) \in \mathscr{B}_1\,,$$

f is said to be $\mathscr{B}_2/\mathscr{B}_1$-**measurable**, and written as $f \in \mathscr{B}_2/\mathscr{B}_1$. Prove the following properties concerning measurability.

- $f \in \mathscr{B}_1/\mathscr{B}_2\,, g \in \mathscr{B}_2/\mathscr{B}_3 \Longrightarrow g \circ f \in \mathscr{B}_1/\mathscr{B}_3$ (**Transition Law** of measurability).
- Suppose $\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_n$ are the σ additive families on, respectively, $S_1, S_2, \cdots, S_n$, and $\pi_i : S_1 \times$

$S_2 \times \cdots S_n \longrightarrow S_i$ is a projection. Then

$$\pi_i \in \mathscr{B}_1 \times \mathscr{B}_2 \times \cdots \times \mathscr{B}_n / \mathscr{B}_i \,, \qquad i = 1, 2, \cdots, n \,.$$

When $\mathscr{B}$ is the σ additive family on $S_1 \times S_2 \times \cdots S_n$,

$$\pi_i \in \mathscr{B} / \mathscr{B}_i \quad (i = 1, 2, \cdots, n) \Longrightarrow \mathscr{B} \supset \mathscr{B}_1 \times \mathscr{B}_2 \times \cdots \times \mathscr{B}_n \,.$$

Accordingly $\mathscr{B}_1 \times \mathscr{B}_2 \times \cdots \times \mathscr{B}_n$ is the smallest σ additive family amongst $\mathscr{B}$'s on $S_1 \times S_2 \times \cdots \times S_n$ that satisfy $\pi_i \in \mathscr{B}/\mathscr{B}_i$ $(i = 1, 2, \cdots, n)$.

(x) Suppose f is a mapping from S_1 into S_2, $\mathscr{B}_i$ is the σ additive family on S_i, and $\mathscr{A}_2$ is certain family subsets of S_2. If

$$E \in \mathscr{A}_2 \Longrightarrow f^{-1}(E) \in \mathscr{B}_1 \,,$$

then $f \in \mathscr{B}_1 / \sigma[\mathscr{A}_2]$.

(xi) Suppose $\mathscr{B}_1$ is a σ additive family on S_1, S_1 is a topological space, and f is a mapping from S_1 onto S_2. When $f \in \mathscr{B}_1 / \mathscr{B}(S_2)$, written simply as $f \in \mathscr{B}_1$, f is **measurable** $\mathscr{B}_1$ or $\mathscr{B}_1$-**measurable**. For any open set G of S_2, if $f^{-1}(G) \in \mathscr{B}_1$, show that $f \in \mathscr{B}_1$. And when $S_2 = \boldsymbol{R}$,

$$f^{-1}\big((\infty, a)\big) \in \mathscr{B}_1 \quad (a \in \boldsymbol{R}^1) \Longrightarrow f \in \mathscr{B}_1 \,.$$

(xii) (**Borel Mapping**) S_1 and S_2 are topological spaces, when f, a mapping from S_1 into S_2, is measurable $\mathscr{B}(S_1)/\mathscr{B}(S_2)$, f is called **Borel measurable mapping** or **Borel mapping**. Show that a continuous map is Borel measurable.

(xiii) (***P*** **Measurable**) Suppose P is a probability measure on S_1, S_2 is a topological space, and f is a mapping from S_1 into S_2. Then when $f \in \mathscr{D}(P)$ (i.e., $f \in \mathscr{D}(P)/\mathscr{B}(S_2)$), f is said to be $\boldsymbol{P}-$ **measurable** or simply **measurable**. If $A \in \mathscr{D}(P)$, show that the indicator function $1_A : S_1 \longrightarrow \boldsymbol{R}^1$ of A is P-measurable.

(xiv) Suppose $\mathscr{B}$, $\mathscr{B}_1$, $\mathscr{B}_2$, $\cdots$, $\mathscr{B}_n$ are, respectively, the σ additive families on $S_1, S_2, \cdots, S_n$; $f_i : S \longrightarrow S_i$ $(i = 1, 2, \cdots n)$; and f is the **product mapping** of $f_1, f_2, \cdots, f_n$, i.e.

$$f : S \longrightarrow S_1 \times S_2 \times \cdots \times S_n \,, \qquad x \longmapsto \big(f_1(x), f_2(x), \cdots, f_n(x)\big) \,.$$

Then prove that

$$f_i \in \mathscr{B}/\mathscr{B}_i \, (i = 1, 2, \cdot, n) \implies f \in \mathscr{B}/\mathscr{B}_1 \times \mathscr{B}_2 \times \cdots \times \mathscr{B}_n$$

[Hint] Show that $\mathscr{B}' = \big\{E \subset S_1 \times S_2 \times \cdots \times S_n \big| f^{-1}(E) \in \mathscr{B}\big\}$ is a σ additive family, and that $E_1 \times E_2 \times \cdots \times E_n$ $(E_i \in \mathscr{B}_i,\ i = 1, 2, \cdots, n)$ is contained in $\mathscr{B}'$.

(xv) (**Lebesgue Extension**) The P-measurable set such that $P(N) = 0$ is called the **null set**. When all the subsets of the P null set are all P-measurable (accordingly P null), P is said to be a complete probability measure. (When P is a complete probability measure on Ω, then the probability space (Ω, P) is called the **complete probability space**). Show that for any probability measure, a complete extension necessarily exists, and moreover that there is a smallest extension (this is called the **Lebesgue extension** of P) amongst all the extensions.

[Hint] Suppose

$$\mathscr{D}(Q) = \left\{ A \middle| \text{there are appropriate } B_1, B_2 \in \mathscr{D}(P) \text{such that } B_1 \subset A \subset B_2, P(B_2 - B_1) = 0 \right\}$$

$$Q(A) = P(B_1) \qquad (\text{with respect to the above } B_1)$$

Then Q is the Lebesgue extension of P.

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