§ 1.6 Independent Real Probability Variable

In this section, all $X, X_1, X_2, \cdots$ are real probability variables.

Theorem 1.23 (Multiplicative Property of Average Value) If $X_1, X_2, \cdots, X_n$ are independent, then

$$E(X_1 X_2 \cdots X_n) = EX_1\, EX_2\, \cdots\, EX_n\,.$$

Proof $n = 1$ is of course true. When $n = 2$,

$$E(X_1X_2) = \sum x_1x_2 P^{(X_1,X_2)}\{(x_1,x_2)\} \qquad \text{(Theorem 1.3 (v))}$$

Here, $\sum$ is the sum over $x_1 \in \Omega^{X_1}$, and $x_2 \in \Omega^{X_2}$. Since X_1 and X_2 are independent,

$$P^{(X_1,X_2)}\{(x_1,x_2)\} = P^{X_1}\{x_1\}P^{X_2}\{x_2\}\,,$$
$$EX_i = \sum x_i P^{X_1}\{x_i\} \qquad \left(\sum \text{ is over } x_i \in \Omega^{X_i}\right).$$

Therefore

$$E(X_1X_2) = EX_1EX_2\,.$$

Since $X_1, X_2, \cdots X_n$ are independent, $(X_1, X_2, \cdots, X_{n-1})$ and X_n are also independent (Theorem 1.18). Hence, using the case of $n = 2$,

$$E\big(X_1X_2\cdots X_{n-1}X_n\big) = E\big(X_1X_2\cdots X_{n-1}\big)EX_n\,.$$

As $X_1, X_2, \cdots, X_{n-1}$ is independent of the independent system $X_1, X_2, \cdots, X_n$ as its subsystem, from the above equation, if the theorem holds with respect to $n-1$, then it also holds for n. ▮

The additive property of average value

$$E(X_1 + X_2 + \cdots + X_n) = EX_1 + EX_2 + \cdots + EX_n$$

holds with respect to any real probability variables. But the multiplicative property is valid only for the independent case. For example, when $X_1 = X_2 = X$,

$$E(X_1X_2) = EX^2 \geqq (EX)^2 = EX_1EX_2\,,$$

and as $EX^2 - (EX)^2 = VX$, the equality in the above does not hold in general.

Theorem 1.24 (**Additive Property of Variance**) If any of the two variables of $X_1, X_2, \cdots, X_n$ are pairwise independent, then

$$V(X_1 + X_2 + \cdots + X_n) = V(X_1) + V(X_2) + \cdots + V(X_n)\,.$$

Proof Let $Y_i = X_i - EX_i$. Then Y_i and $Y_j (i \neq j)$ are independent. Moreover

$$V(X_i) = E(Y_i^2)\,, \qquad V(X_1 + X_2 + \cdots + X_n) = E\big((Y_1 + Y_2 + \cdots Y_n)^2\big)\,.$$

From the multiplicative property of average value,

$$E(Y_iY_j) = EY_iEY_j = 0\,, \qquad i \neq j\,.$$

Hence

$$E\big((Y_1 + Y_2 + \cdots + Y_n)^2\big) = \sum_i E(Y_i^2) + 2\sum_{i<j} E(Y_iY_j) = \sum_i E(Y_i^2)\,,$$

that is

$$V(X_1 + X_2 + \cdots + X_n) = V(X_1) + V(X_2) + \cdots + V(X_n)\,. \quad \blacksquare$$

If $X_1, X_2, \cdots, X_n$ are independent, then since the assumptions of the above theorem hold, the addictive property of variance is obtained. The above theorem asserts that even under the condition weaker than the independence of $X_1, X_2, \cdots, X_n$, the same can be said. But it is not without any conditions. In fact, suppose $X_1 = X_2 = \cdots = X_n = X$, then

$$V(X_1 + X_2 + \cdots + X_n) = n^2 V(X)\,, \qquad V(X_1 + X_2 + \cdots + X_n) = nV(X)\,.$$

If X is a real probability variable, for any $t \in (0, 1)$, t^X is also a real probability variable. The function of t

$$g^X(t) = E\big(t^X\big)\,, \qquad t \in (0, 1)$$

is said to be the **generating function** of X. From Theorem 1.3 (v),

$$g^X(t) = \sum_{x \in \Omega^X} t^x\, P^X\{x\}\,.$$

By this formula, the probability law P^X of X is determined by the generating function $g^X 9t)$. Indeed, suppose the points of Ω^X are sorted on size, $x_1 < x_2 < \cdots < x_n$, then the order of $P^X\{x_k\}$ is determined in the following:

$$P^X\{x_1\} = \lim_{t\downarrow 0} t^{-x_1} g^X(t)\,,$$
$$P^X\{x_k\} = \lim_{t\downarrow 0} t^{-x_k} \left(g^X(t) - \sum_{i=1}^{k-1} t^{x_i} P^X\{x_i\} \right)\,.$$

If $X_1, X_2, \cdots X_n$ are independent, then since $t^{X_1}, t^{X_2}, \cdots, t^{X_n}$ are independent, by the multiplicative property of average value, the following theorem is obtained.

Theorem 1.25 (**Multiplicative property of generating function**) Suppose $X_1, X_2, \cdots X_n$ are independent, and let $X = X_1 + X_2 + \cdots + X_n$. Then

$$g^X(t) = g^{X_1}(t) g^{X_2}(t) \cdots g^{X_n}(t)\,.$$

Example 1.3 $A_1, A_2, \cdots, A_n$ are independent, and $P(A_i) = p$, independent or i. Suppose N is the number of occurrences of $A_1, A_2, \cdots, A_n$. Show that

$$P(N = k) = \binom{n}{k} p^k (1-p)^{n-k}\,, \qquad k = 0, 1, 2, \cdots, n\,.$$

Let $e_i(\omega)$ be the indicator function of A_i. Then

$$g^{e_i}(t) = t^1 p(e_i = 1) + t^0 p(e_i = 0) = tp(A_i) + p(A_i^c) = tp + (1-p)\,.$$

Express N as $\{e_i\}$:

$$N = e_1 + e_2 + \cdots e_n\,.$$

Since $\{e_i\}$ is independent, from the independence property of generating function

$$g^X(t) = \prod_{i=1}^{n} g^{e_i}(t) = \big(tp + (1-p)\big)^n = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} t^k\,.$$

Hence, $P(N = k) = P^N\{k\} = \binom{n}{k} p^k (1-p)^{n-k}$.

Example 1.4 A dice is rolled thrice. Find the probabilities p_9 and p_{10} for which the sums of the points are 9 and 10.

Let the points be X_1, X_2, X_3, and their sum be S.

$$g^{X_i}(t) = \frac{1}{6}\big(t + t^2 + \cdots + t^6\big) = \frac{1}{6}(1-t)^{-1}(1-t^6)t\,.$$

From Theorem 1.22, since X_1, X_2, X_3 are independent, by the multiplicative property of generating function,

$$
\begin{aligned}
g^X(t) &= \prod_{i=1}^{3} g^{X_i}(t) = 6^{-3}(1-t)^{-3}(1-t^6)^3 t^3 \\
&= 6^{-3}\left(1 + 3t + \frac{3\cdot 4}{2}t^2 + \frac{4\cdot 5}{2}t^3 + \frac{5\cdot 6}{2}t^4 + \cdots\right)\left(1 - 3t^6 + 3t^{12} - t^{18}\right)t^3\,,
\end{aligned}
$$

$$
\begin{aligned}
p_9 &= \text{“}t^9\text{'s coefficient of } g^S(t)\text{”} = 6^{-3}\left(\frac{7\cdot 8}{2} - 3\right) = \frac{25}{216}\,, \\
p_{10} &= \text{“}t^{10}\text{'s coefficient of } g^S(t)\text{”} = 6^{-3}\left(\frac{8\cdot 9}{2} - 9\right) = \frac{27}{216}\,,
\end{aligned}
$$

Exercise 1.6 Using the multiplicative property of average value, prove that “if $X_1, X_2, \cdots, X_n$ are independent, $EX_i = 0$, $EX_i^2 = 1\,, (i = 1, 2, \cdots, n)$, then

$$
E\big((X_1 + X_1X_2 + X_1X_2X_3 + \cdots + X_1X_2X_3\cdots X_n)^2\big) = n\,.\text{”}
$$