Postscript

The literature on probability theory has expanded rapidly over the last twenty years or so. It was not within my scope to cover every aspect of probability theory in this lecture course. What I did was to select a portion of probability theory and to provide an introduction to it. For you who may want to further your research study, I would like to cite a few references chapter by chapter.

Although the content of Chapter 1 is essentially at the level of high school mathematics, it is viewed from the perspective of modern probability theory.

In Chapter 2, I described measure theory for what is immediately needed in probability theory.

In Chapter 3, I showed how the finite trials explained in Chapter 1 may become when they are treated in general. Here, I expounded it on the basis of probability measure when it is restricted to that which is completely separable. For applications, this exposition would be sufficient. Also, it is convenient to think about conditional probability measure on the sample space. Though there are many other advantages but I later realized that at times, there are also unreasonable points. Even for restriction, it may be better to follow Kolmogorov by requiring only completeness. It is Doob who defined conditional probability with respect to the family of σ additive measures, whereas Kolmogorov defined it with respect to partition. The latter is more precise and has a better match with intuition. But depending on the problems, formulation of deduction may be simpler by using the familty of σ addivtive measures. The upshot is that freely using both approaches is desirable.

The sum of independent random variables in Chapter 4 is now a classic. In this lecture course, I restricted it to real-valued random variables and introduced a portion of the theory. The reference books I recommend for this area are

> B. V. Gnedenko and A. N. Kolmogorov: Limit distributions for sums of independent random variables, Addison-Wesley, 1954.
>
> W. Feller: An introduction to probability theory and its applications, vol 2, John Wiley, 1966.

Chapter 5 is on the theory of stochastic processes. Though I cannot possibly introduce each and every aspect, this theory now constitutes most of the probability theory. Despite the fact, it is quite regrettable that covering even the fundamental things takes up too many pages, and I cannot help but to cover only the important ones. In § 5.1 up to § 5.4, I have explained in detail the basic things for the purpose of using them subsequently. However, as the number of pages left for writing has gotten fewer, the explanation for the rest of the chapter may not be sufficient. For the theory of martingale in § 5.5 and § 5.6, it is not an overstatement to say that I owe almost all of it to J. L. Doob. In these two sections, I explained the theory of Doob. For this area, research of French school centered around P. Meyer is remarkable. As an example, Probabilités et potential, Hermann, 1966

contains many important results after Doob.

One can probably identify the Gaussian system in § 5.7 as a theory of a Gaussian distribution of infinite dimension. In §, I covered the classical results of the Wiener process, which is the most basic among all stochastic processes. It is the foundation for the theory of stochastic differential equation covered in the following section. For an excellent reference book on the Wiener process, I recommend

Takeyuki Hida, Brownian motion, Iwanami Shoten, 1975.

The content of this book covers a wide area.

Polynomial and Poisson configurations are being used frequently in applications. I do not think that there is an explanation as detailed as this lecture.

The additive process in § 5.10 is based on a famous book by P. Lévy:

Theorie de l'addition des variables aléatoires, Gauthier-Villars, 1937.

I need to write many pages in order to give a rigorous formulation and proof of Lévy's decomposability theorem in his book. Research in the 1930s on the theory of independent random variables, which was explained in Chapter 4, has constructed the theory of discrete-time process with an eye on the additive process (for continuous-time random variables). Lévy approached the continuous-time process directly and constructed a thoroughly clear theory. His writing is very intuitive, but the formulation as well as the proofs are not necessarily as clear.

Given the measure-theoretic foundation that is convenient for constructing the theory of stochastic processes by Kolmogorov, Doob, and others, I made it a point to explain Lévy's thinking along this line. In Lévy's book cited above, as applications of this theorem, many interesting results can be obtained. This lecture course introduced merely a portion.

§ 5.12, § 5.13, and § 5.18 are about the introductory theory concerning the Markov process. Please refer to, for example,

K. Itô and H. P. McKean: One-dimensional diffusion and their sample functions, Springer, 1969.

And,

E. B. Dynkin: Markov processes, I, II, Academic Press, 1966

can be said to be the grand compilation of the achievements thus far in researching the Markov process.

From § 5.14 through § 5.16, I covered the classical theory of stochastic differential equation founded on the Wiener process, with some tidying up to make it look a little bit modern. This is about just right for applications. But to render the theory more crystal clear and natural, as suggested by Doob, one must use martingale instead of the Wiener process to define stochastic integral. As a result, it became very natural to be able to define stochastic differentiation. I want to recommend a good book that gives a structured treatment of this new theory and its applications:

Shinzo Watanabe, Stochastic Differential Equations, Sangyo Tosho, 1975.

Furthermore, though I did not touch on Markov's statistical mechanics, it goes without saying that the relationships with ergodic theory, Markov process, and potential theory, are extremely important too.

Christopher Ting
Translation Version 1.0
Date: 23 May, 2017